Multimedia course

CONTINUUM MECHANICS FOR ENGINEERS

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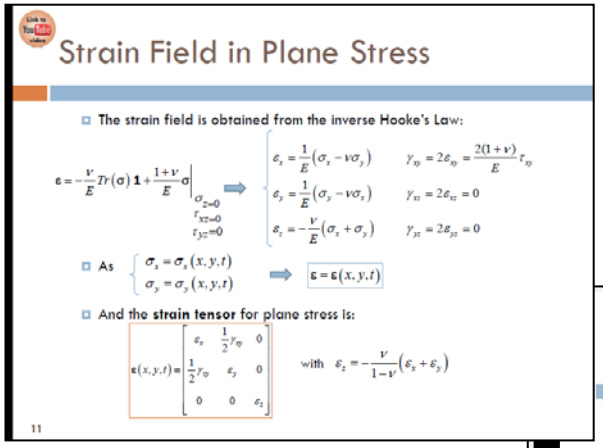

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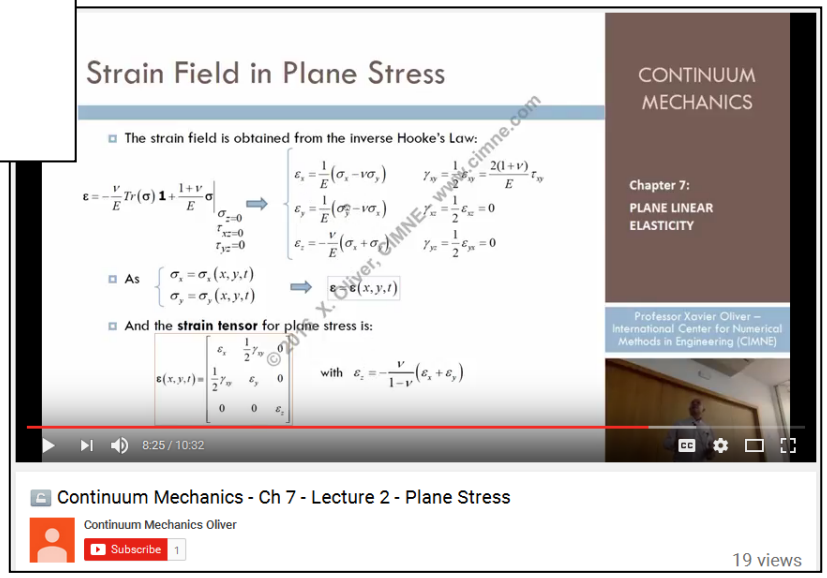

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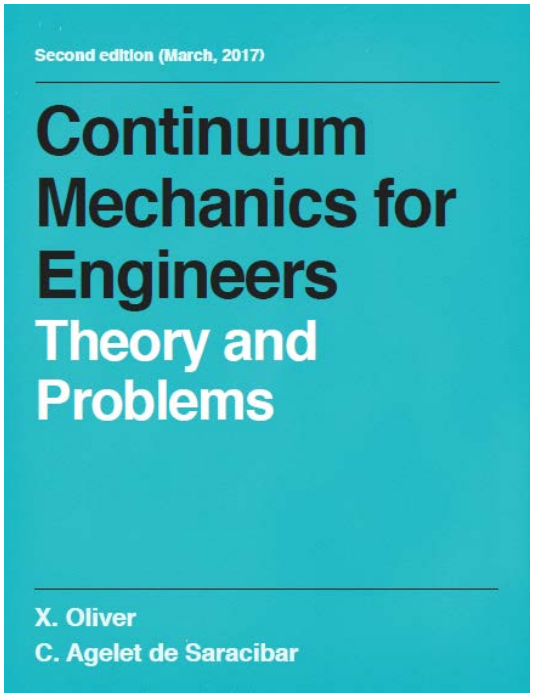

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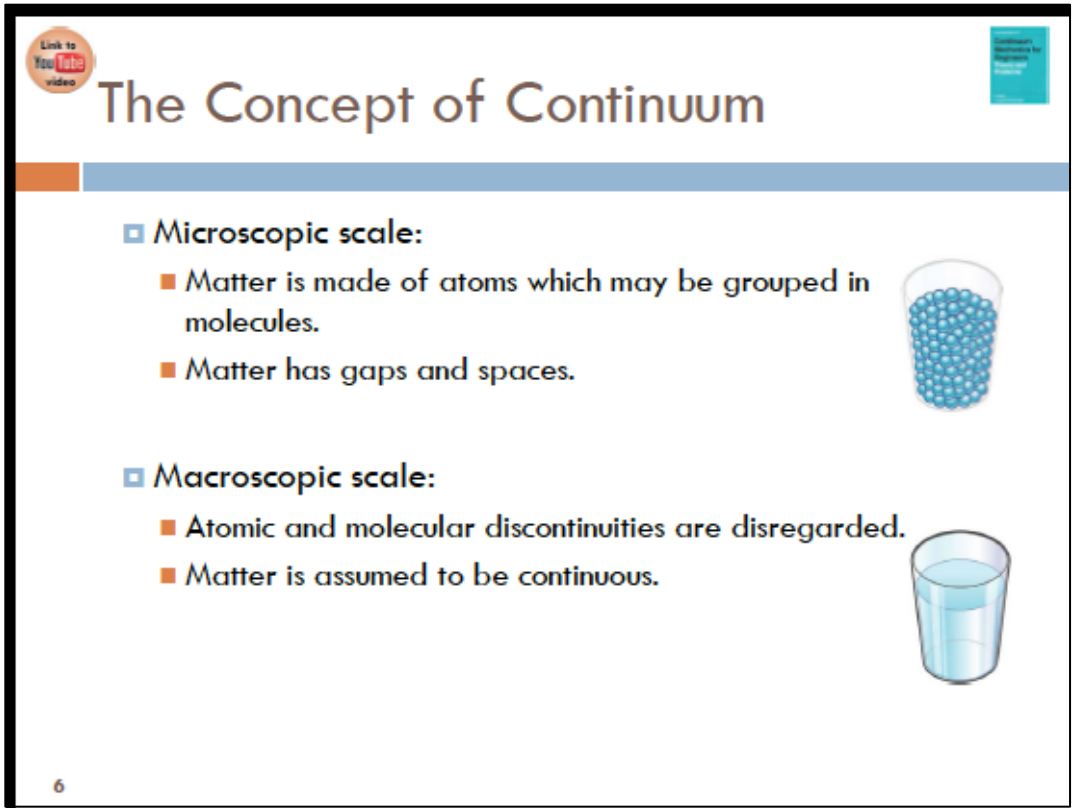

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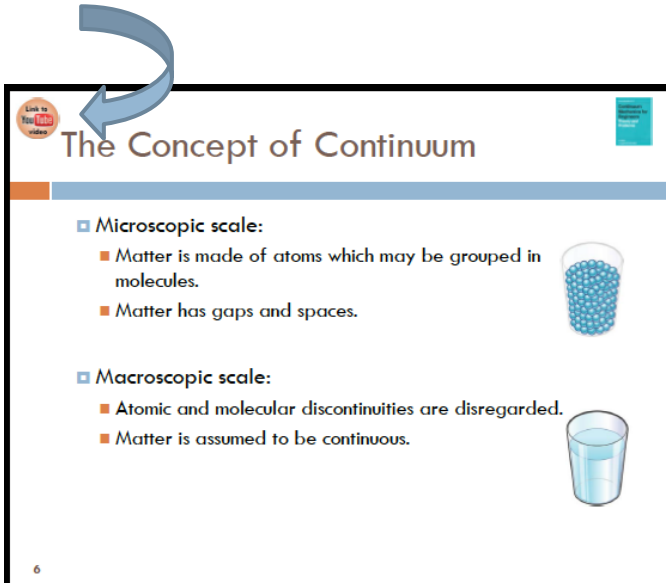

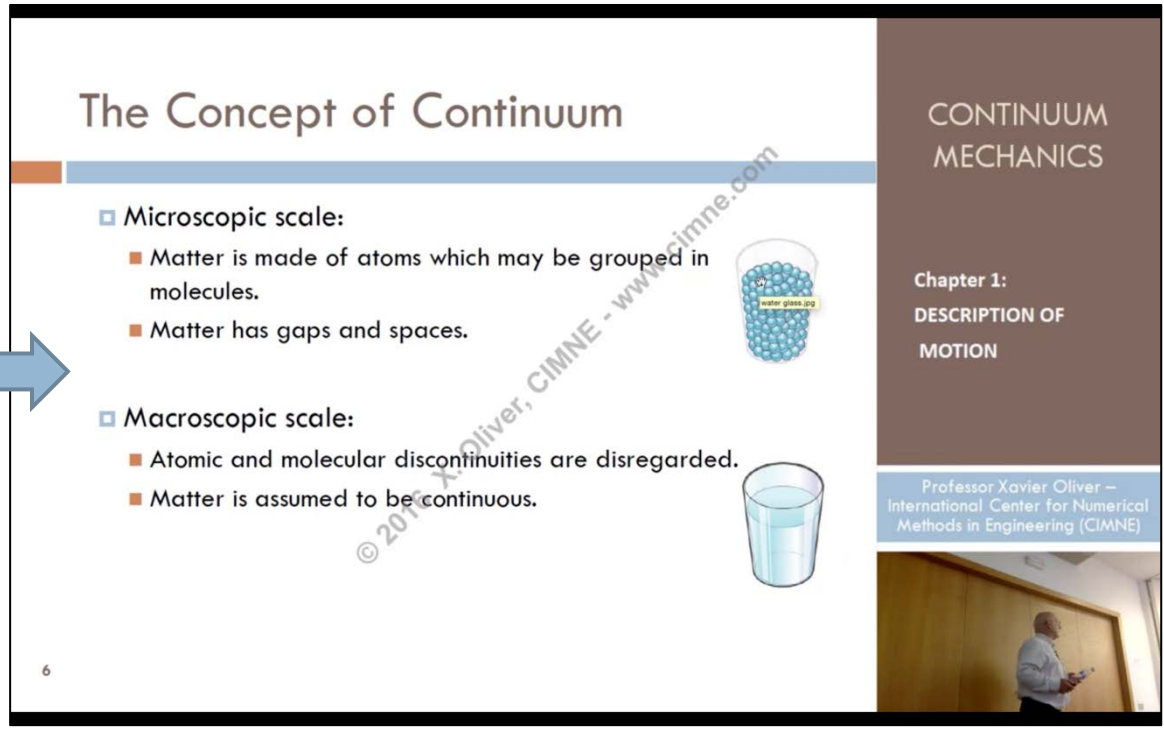

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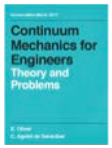
3 To reach the book-section dealing with the slide, click on the **link-to-book** icon 

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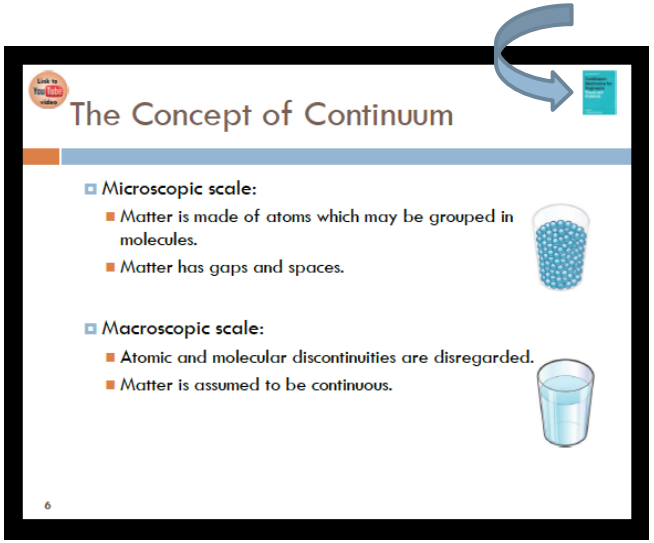

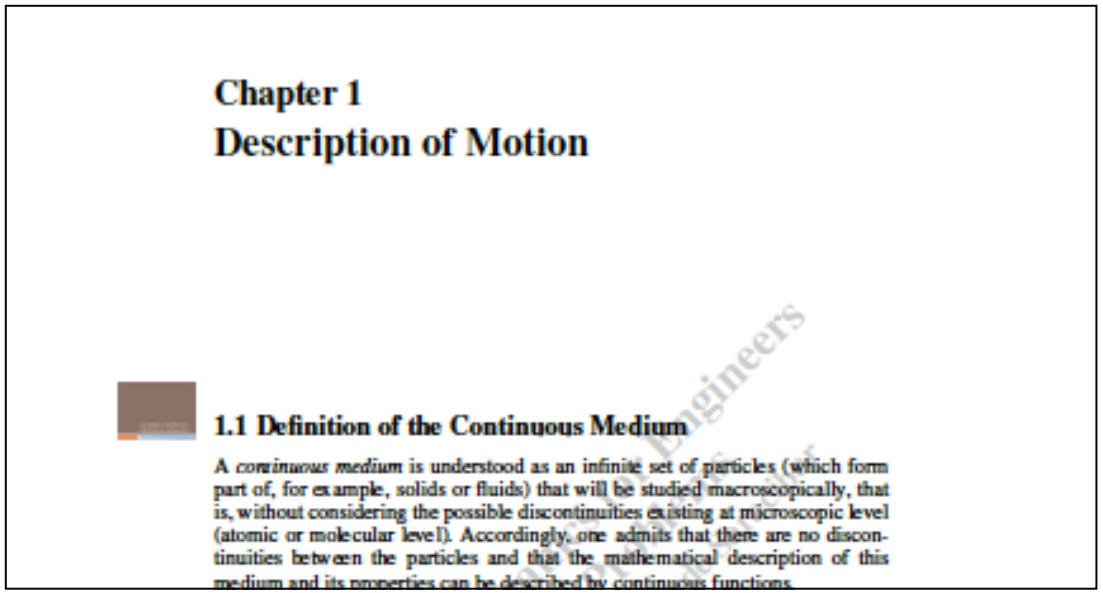

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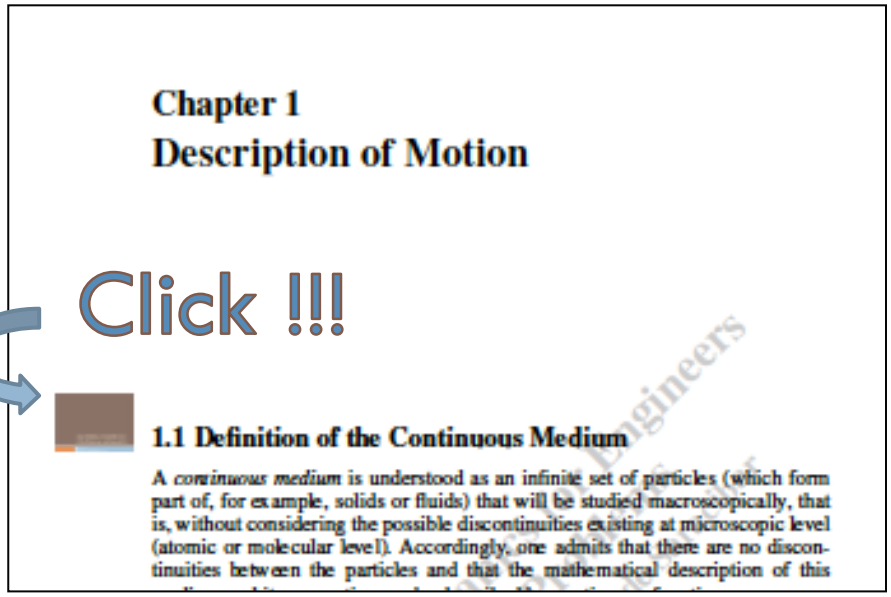

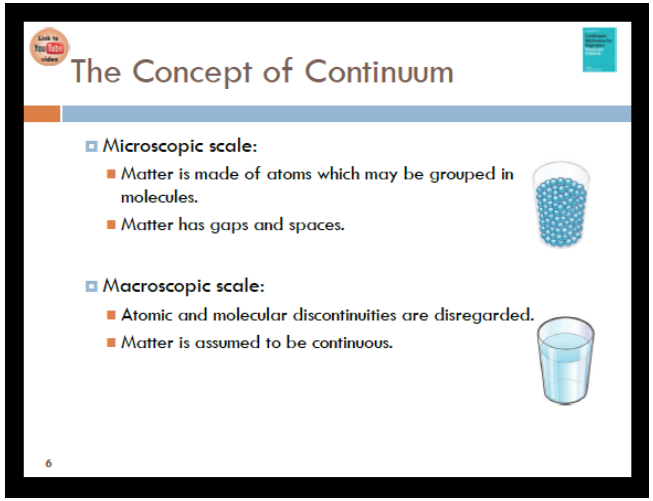

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... and the past students of my courses on Continuum Mechanics ...

Thanks for your contribution !!!!

CH.10. FLUID MECHANICS

Multimedia Course on Continuum Mechanics

Overview

- Governing Equations
  - Newtonian Fluids
  - Barotropic fluids

- Hydrostatics. Fluids at rest
  - Hydrostatic problem
  - Archimedes´ Principle
  - Equilibrium of Floating Solids

- Barotropic Perfect Fluids
  - Fluid Mechanics Equations
  - Bernoulli's Trinomial
  - Steady State Solution
  - Transient State Solution

Lecture 1

Lecture 2

Lecture 3

Lecture 4

Lecture 5

Lecture 6

Lecture 7

Lecture 8

Lecture 9

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Overview (cont'd)

- Newtonian Viscous Fluids
  - Navier-Stokes Equations
  - Energy Equation
  - Reduced System of Equations
  - Physical Interpretations

Lecture 10 Link to YouTube video

- Reduced System of Equations for Particular Cases

Lecture 11 Link to YouTube video

- Boundary Conditions
  - BC in velocities
  - BC in pressures
  - Mixed BC
  - BC on free surfaces

Lecture 12 Link to YouTube video

- Laminar and Turbulent Flows

Lecture 13 Link to YouTube video

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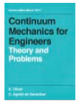
10.1. Governing Equations

Ch.10. Fluid Mechanics

Link to YouTube video



Reminder – Governing Eqns.

- **Balance equations** of the thermo-mechanical problem:

$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0$	Conservation of Mass. Continuity Equation.	1 eqn.
$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$	Linear Momentum Balance. Cauchy's Motion Equation.	3 eqns.
$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$	Angular Momentum Balance. Symmetry of Cauchy's Stress Tensor.	3 eqns.
$\rho \dot{u} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \nabla \cdot \mathbf{q}$	Energy Balance. First Law of Thermodynamics.	1 eqn.
$-\rho(\dot{u} - \theta \dot{s}) + \boldsymbol{\sigma} : \mathbf{d} \geq 0$ $-\frac{1}{\rho \theta^2} \mathbf{q} \cdot \nabla \theta \geq 0$	Clausius-Planck Inequality. Heat flux Inequality.	Second Law of Thermodynamics. 2 restrictions

8 PDE + 2 restrictions

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Newtonian Fluids

- **Constitutive equations** of the thermo-mechanical problem in a **Newtonian fluid:**

12 PDE

$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda Tr(\mathbf{d})\mathbf{1} + 2\mu\mathbf{d}$	Thermo-Mechanical Constitutive Equations.	6 eqns.
$s = s(\mathbf{d}, \rho, \theta)$	Entropy Constitutive Equation.	1 eqn.
$\mathbf{q} = -K\,\nabla\theta$	Thermal Constitutive Equation. Fourier's Law of Conduction.	3 eqns.
$u = f(\rho, \theta)$ (Caloric) $F(\rho, p, \theta) = 0$ (Kinetic)	State Equations.	2 eqns.

- Grand total of 20 PDE with 20 unknowns:

$$\rho \to 1, \quad \mathbf{v} \to 3, \quad \boldsymbol{\sigma} \to 9, \quad u \to 1, \quad \mathbf{q} \to 3, \quad \theta \to 1, \quad s \to 1, \quad p \to 1$$

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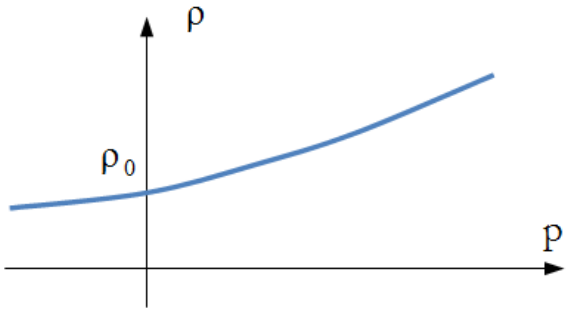
Barotropic Fluids

- A **barotropic fluid** is characterized by the kinetic state equation:

$$F(\rho, p, \cancel{\theta}) = 0 \quad \Rightarrow \quad \rho = \rho(p)$$



- The uncoupled mechanical problem becomes:

$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0$	Conservation of Mass. Continuity Equation.	1 eqn.
$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$	Linear Momentum Balance. First Cauchy's Motion Equation.	3 eqns.
$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda Tr(\mathbf{d})\mathbf{1} + 2\mu\mathbf{d}$	Thermo-Mechanical Constitutive Equations.	6 eqns.
$\rho = \rho(p)$	Kinetic State Equation	1 eqn.

- 11 scalar unknowns: $\rho, \mathbf{v}, \boldsymbol{\sigma}, p$. (Considering the symmetry of Cauchy stress tensor, $\boldsymbol{\sigma}$, will have 6 unknowns).

10.2. Hydrostatics. Fluids at Rest

Ch.10. Fluid Mechanics

Hydrostatic stress state vs. Hydrostatic problem

- **Uniform velocity,** $\mathbf{v}(\mathbf{x},t) \equiv \mathbf{v}(t) \Rightarrow \nabla\mathbf{v} = \nabla \otimes \mathbf{v} = \mathbf{v} \otimes \nabla = \mathbf{0}$

$$\mathbf{d} = \nabla^S \mathbf{v} = \frac{1}{2}\left[\mathbf{v} \otimes \nabla + \nabla \otimes \mathbf{v}\right] = \mathbf{0}$$

Thus, $\boldsymbol{\sigma} = -p\mathbf{1} + \lambda \underbrace{Tr(\mathbf{d})}_{=\mathbf{0}}\mathbf{1} + 2\mu \underbrace{\mathbf{d}}_{=\mathbf{0}} \Rightarrow \boxed{\boldsymbol{\sigma} = -p\mathbf{1} \Rightarrow \overline{p} = p}$ ⇒ HYDROSTATIC STRESS SATE

- **Uniform and stationary velocity,** $\mathbf{v}(\mathbf{x},t) \equiv cnt$:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{0}$$

Thus, $\boxed{\boldsymbol{\sigma} = -p_0\mathbf{1}}$ ⇒ $Tr(\boldsymbol{\sigma}) = -3p_0$ ⇒ $\boxed{\overline{p} = p = p_0}$ HYDROSTATIC CASE

- **Fluid at rest,** $\mathbf{v}(\mathbf{x},t) \equiv cnt = \mathbf{0}$. A particular hydrostatic case (where the name comes from)

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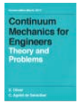
Link to YouTube video



Hydrostatic Problem

- A **hydrostatic problem** ($\mathbf{v}(\mathbf{x},t) \equiv cnt$) is characterized by:

$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0 \Rightarrow \rho(\mathbf{X},t) = \rho_0(\mathbf{X})$	Conservation of Mass. Continuity Equation.	1 eqn.
$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \Rightarrow \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \mathbf{0}$	Linear Momentum Balance. First Cauchy's Motion Equation.	3 eqns.
$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda Tr(\mathbf{d})\mathbf{1} + 2\mu\mathbf{d} \Rightarrow \boldsymbol{\sigma} = -p_0\mathbf{1}$	Thermo-Mechanical Constitutive Equations.	6 eqns.

- Substituting the constitutive and the continuity eqn. into the Cauchy eqn.:

$$\boldsymbol{\sigma} = -p_0\mathbf{1} \Rightarrow \nabla \cdot \boldsymbol{\sigma} = \nabla \cdot (-p_0\mathbf{1}) = -\nabla p_0 \rightarrow \begin{cases} -\nabla p_0 + \rho_0 \mathbf{b} = \mathbf{0} \\ -\dfrac{\partial p_0}{\partial x_i} + \rho_0 b_i = 0 \qquad i \in \{1,2,3\} \end{cases}$$

FUNDAMENTAL EQUATION OF HYDROSTATICS

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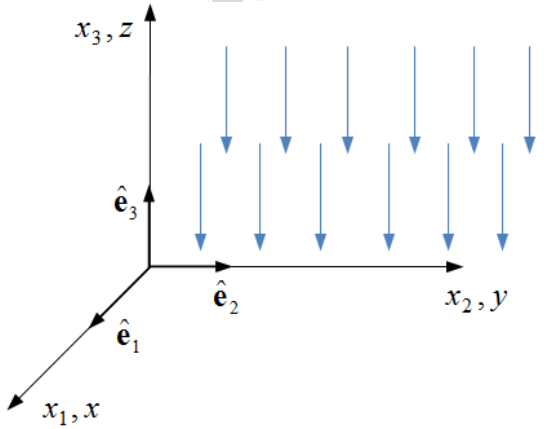
Gravity forces. Triangular pressure distribution

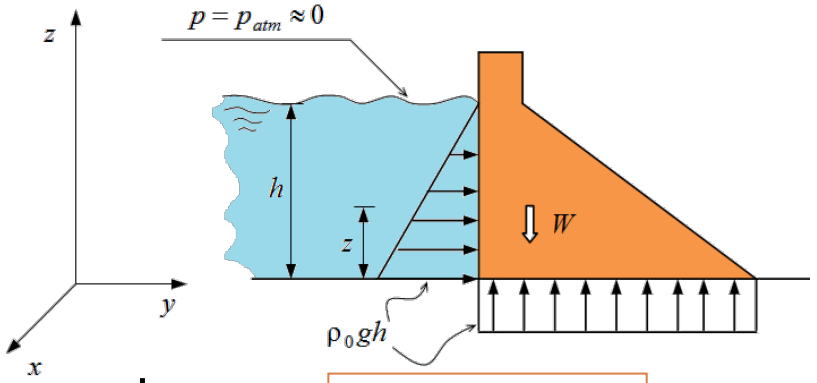
- For a fluid subjected to gravity forces,

$$\begin{cases} -\nabla p_0 + \rho_0 \mathbf{b} = \mathbf{0} \\ -\dfrac{\partial p_0}{\partial x_i} + \rho_0 b_i = 0 \qquad i \in \{1,2,3\} \end{cases} \qquad \mathbf{b}(\mathbf{x},t) = \begin{Bmatrix} 0 \\ 0 \\ -g \end{Bmatrix}$$



the momentum eq. can be written as:

$$\begin{cases} -\dfrac{\partial p_0(x,y,z)}{\partial x} = 0 \xrightarrow{\int dx} p_0(x,y,z) \equiv p_0(y,z) \\ -\dfrac{\partial p_0(y,z)}{\partial y} = 0 \xrightarrow{\int dy} p_0(y,z) \equiv p_0(z) \\ -\dfrac{dp_0(z)}{dz} - \rho_0 g = 0 \xrightarrow{\int dz} p_0 = -\rho_0 g z + C \end{cases}$$



- If the surface pressure is considered zero, then:

$$p_0\big|_{z=h} = 0 \Rightarrow -\rho_0 g\, h + C = 0 \Rightarrow C = \rho_0 g\, h \Rightarrow p_0 = \underbrace{\rho_0 g\,(h - z)}_{\text{Triangular pressure distribution}}$$

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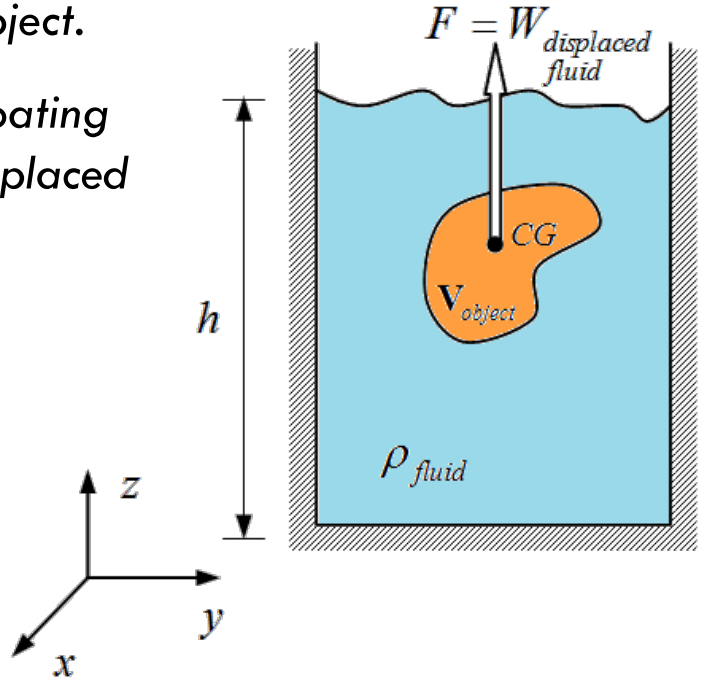
Link to YouTube video

Archimedes´ Principle

- *Any fluid applies a **buoyant force (up-thrust)** to an object that is partially or completely immersed in it.*
  - *The **magnitude** of the buoyant force is equal to the **weight of the fluid** displaced by the object.*
  - *The **resultant** of the buoyant force on a floating object acts at the **center of mass** of the displaced fluid (center of buoyancy).*



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Archimedes´ Principle - Proof

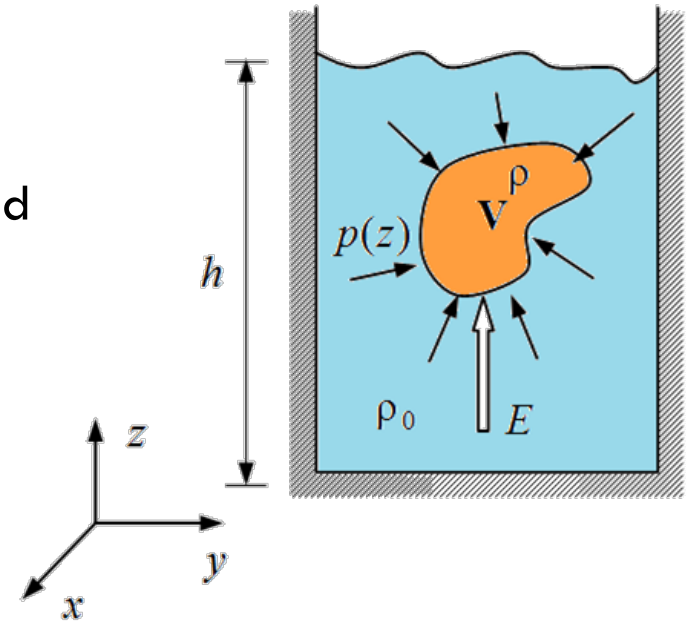
- Consider a solid with a volume V and density ρ within a fluid in a **hydrostatic case.** Then,
  - The **traction vector** on the solid boundary :

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} = -p_0 \mathbf{1} \cdot \mathbf{n} = -p_0 \mathbf{n}$$

$$p_0(z) = \rho_0 g \ (h - z)$$

  - The **resultant force** exerted by the fluid **on the solid :**

$$\mathbf{R} = \int_{\partial V} \mathbf{t} \, dS = \int_{\partial V} -p_0(z)\mathbf{n} \, dS$$



⇨ **R** only depends on the hydrostatic pressure distribution on the boundary of the solid

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Archimedes´ Principle (first part proof)

$$\mathbf{R} = \int_{\partial V} \mathbf{t}\, dS = \int_{\partial V} -p_0(z)\mathbf{n}\, dS$$

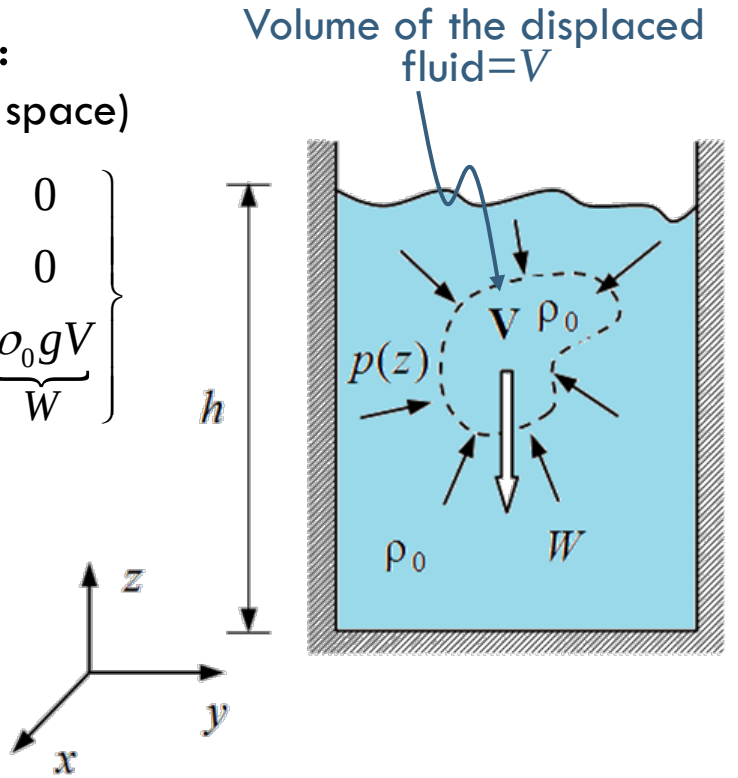
$$-\nabla p_0 + \rho_0 \mathbf{b} = \mathbf{0}$$

- Consider the same fluid **without the solid** in it, and replaced by fluid . Then,
  - Pressures on the **boundary** of the "replacing" fluid are the same than in the immersed solid case (and, therefore, the resulting force, $\mathbf{R}$)
  - The divergence theorem can be applied:
    (The pressure distribution is now continuous in space)

$$\mathbf{R} = \int_{\partial V} -p_0 \mathbf{n} dS = \int_V \underbrace{-\nabla p_0}_{-\rho_0 \mathbf{b}}\, dV \equiv -\rho_0 \underbrace{\begin{Bmatrix} 0 \\ 0 \\ -g \end{Bmatrix}}_{\mathbf{b}} \underbrace{\int_V dV}_{V} = \begin{Bmatrix} 0 \\ 0 \\ \underbrace{\rho_0 g V}_{W} \end{Bmatrix}$$

- Finally, $\mathbf{R} = E\,\hat{\mathbf{e}}_z = W\,\hat{\mathbf{e}}_z \Rightarrow E = W$

Up-thrust on the body (E) = weight of the fluid displaced by the body (W)



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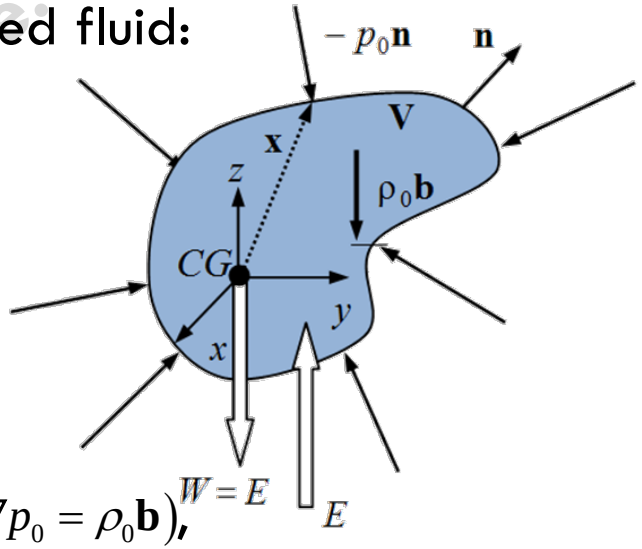
Archimedes´ Principle (second part proof)

$$-\nabla p_0 + \rho_0 \mathbf{b} = \mathbf{0}$$

- Consider the moment of the up-thrust forces at the center of mass (center of gravity, CG) of the volume of displaced fluid:



Divergence Theorem

$$\mathbf{M}_E^G = \int_{\partial V} \mathbf{x} \times (-p_0 \mathbf{n})\, dS = \int_V \mathbf{x} \times (-\underbrace{p_0 \nabla}_{=\nabla p_0})\, dV =$$

$$= -\int_V (\mathbf{x} \times \nabla p_0)\, dV$$

- Substituting the fundamental eq. of hydrostatics $(\nabla p_0 = \rho_0 \mathbf{b})$,

$$\mathbf{M}_E^G = -\int_V (\mathbf{x} \times \rho_0 \mathbf{b})\, dV = -\mathbf{M}_W^G = \mathbf{0}$$

$= \mathbf{M}_W^G$ ⇨ Moment of the weight of the displaced fluid with respect to its center of gravity (by definition it must be zero)

⇨ The up-thrust force, **E**, passes through the CG of the volume of the displaced fluid (center of buoyancy).

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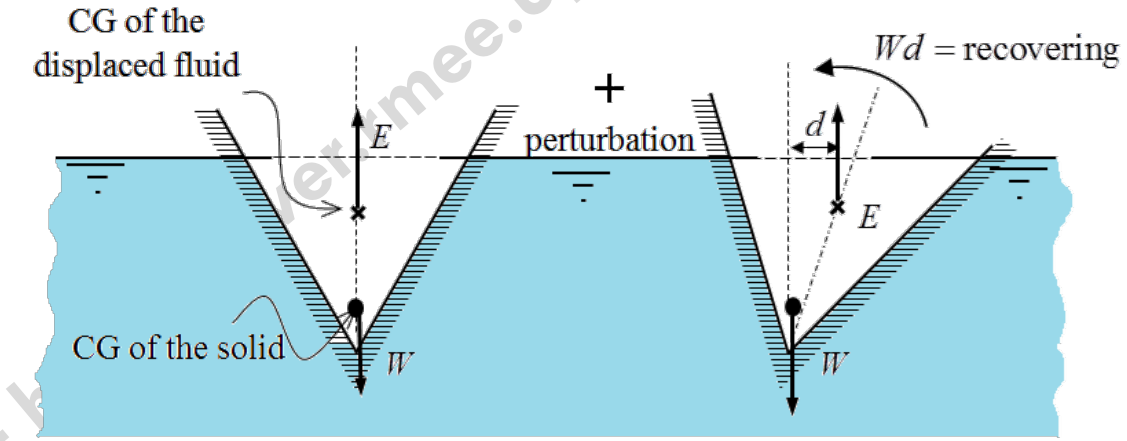
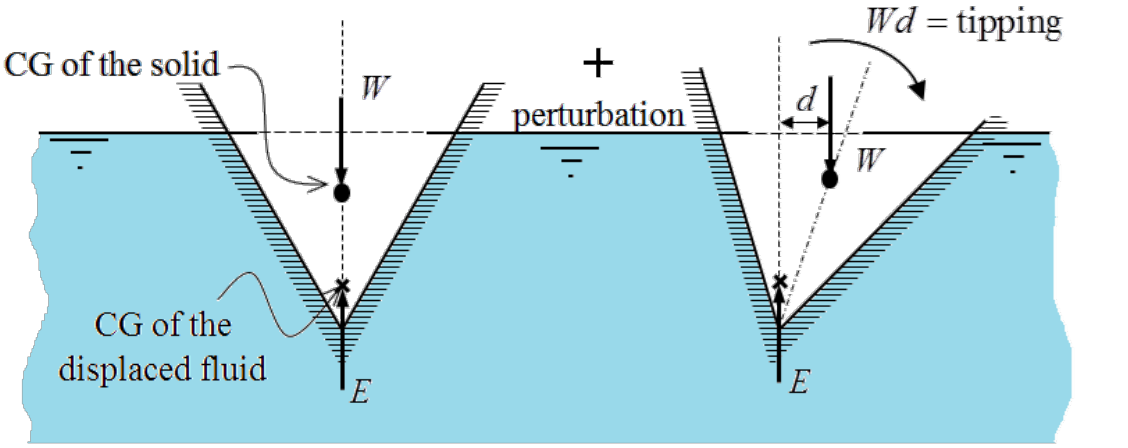
Equilibrium of Floating Solids

- The equilibrium can be:
  - **Stable**: the solid's *CG* is <u>below</u> the center of buoyancy (*CG* of the displaced fluid).
  - **Unstable**: the solid's *CG* is <u>above</u> the center of buoyancy (*CG* of the displaced fluid).





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10.3. Fluid Dynamics. Barotropic Perfect Fluids

Ch.10. Fluid Mechanics

Link to YouTube video

Barotropic Perfect Fluids

- A **perfect fluid** is a Newtonian fluid with null viscosity, $\mu = \lambda = 0$:

$$\boldsymbol{\sigma} = -p\mathbf{1} + \cancel{\lambda} Tr(\mathbf{d})\mathbf{1} + 2\cancel{\mu}\mathbf{d} \quad \Rightarrow \quad \boldsymbol{\sigma} = -p\mathbf{1}$$

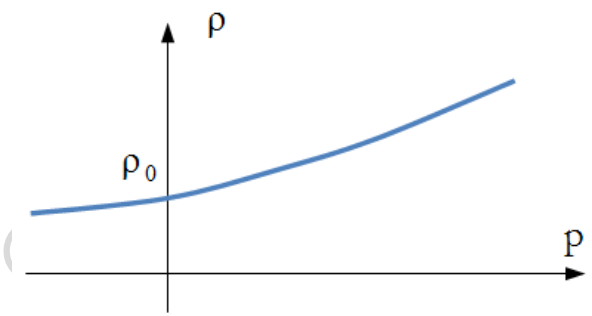
hydrostatic stress state

  - Therefore,

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} = -\nabla p \\ \boldsymbol{\sigma} : \mathbf{d} = -p\mathbf{1} : \mathbf{d} = -pTr(\mathbf{d}) \end{cases}$$

- In a **barotropic fluid** temperature does not intervene in the kinetic state equation:

$$F(\rho, p, \cancel{\theta}) = 0 \quad \Rightarrow \quad \rho = \rho(p)$$



REMARK

Do not confuse a hydrostatic stress state (spherical stress tensor) with a hydrostatic flow regime (null or uniform velocity).

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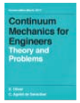

Barotropic Perfect Fluids: Field Equations

- The **mechanical** problem for a **barotropic perfect fluid**:

$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0$	Conservation of Mass. Continuity Equation.	1 eqn.
$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \Rightarrow -\nabla p + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$	Linear Momentum Balance. Euler's Equation.	3 eqns.
$F(\rho, p, \cancel{\theta}) = 0 \Rightarrow \rho = \rho(p)$	Kinetic State Equation	1 eqn.

- 5 scalar unknowns: ρ, $\mathbf{v}$, p.

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Barotropic Perfect Fluids: Field Equations

- The **thermal** problem for a **barotropic perfect fluid**:

$\mathbf{q} = \mathbf{q}(\theta) = -K\ \nabla\theta$	Thermal Constitutive Equation. Fourier's Law of Conduction.	3 eqns.
$\rho\dot{u} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \nabla\cdot\mathbf{q}$ $\Rightarrow\ \rho\dot{u} = -p\nabla\cdot\mathbf{v} + \rho r + K\nabla^2\theta$	Energy Balance. First Law of Thermodynamics.	1 eqn.
$u = f(\rho, \cancel{\mathbf{v}}, \theta, \cancel{\mathbf{g}}) \Rightarrow\ u = u(\rho,\theta)$	Caloric State Equation	1 eqn.

- Once the mechanical problem is solved, the thermal problem can be calculated as there are 5 scalar unknowns: $u \to 1,\ \ \mathbf{q} \to 3,\ \ \theta \to 1$

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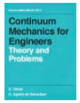
Link to YouTube video



Bernoulli's Trinomial

- Consider a barotropic fluid with **potential body forces:**

$$\underbrace{\phi(\mathbf{x},t)}_{\text{Body forces potential}} = gz \rightarrow \mathbf{b}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t) = -\begin{bmatrix} \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{bmatrix}^T = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

- And, consider the following lemmas:
  - ***Lemma 1.*** For a barotropic fluid there exists a function $\mathbb{P}(\mathbf{x},t) = \hat{\mathbb{P}}(p(\mathbf{x},t))$ which satisfies:

$$\boxed{\nabla p = \rho \nabla \mathbb{P}}$$

Proof : $\mathbb{P}(p) \equiv \int_0^p \frac{1}{\rho(\tilde{p})} d\tilde{p} \Rightarrow \nabla\mathbb{P} = \frac{1}{\rho(p)} \nabla p$

  - ***Lemma 2.*** The convective term of the acceleration can be written as:

$$\boxed{\mathbf{v}\cdot\nabla\mathbf{v} = 2\boldsymbol{\omega}\times\mathbf{v} + \nabla\left(\frac{1}{2}\mathrm{v}^2\right)}$$

Where $2\boldsymbol{\omega} = \nabla\times\mathbf{v}$ is the vorticity vector.

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Bernoulli's Trinomial

- Taking the Euler equation and substituting,

$$-\nabla p + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \rightarrow -\frac{1}{\rho}\nabla p + \mathbf{b} = \underbrace{\frac{d\mathbf{v}}{dt}}_{\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}}$$

$$\frac{1}{\rho}\nabla p = \nabla \mathbb{P} \quad ; \quad \mathbf{b} = -\nabla \phi$$

$$\mathbf{v}\cdot\nabla\mathbf{v} = 2\boldsymbol{\omega}\times\mathbf{v} + \nabla\left(\tfrac{1}{2}\mathrm{v}^2\right)$$

$$\Rightarrow -\nabla \mathbb{P} - \nabla\phi = \frac{\partial \mathbf{v}}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{v} + \nabla\left(\tfrac{1}{2}\mathrm{v}^2\right)$$

- Rearranging,

$$-\nabla\left[\mathbb{P} + \phi + \tfrac{1}{2}\mathrm{v}^2\right] = \frac{\partial \mathbf{v}}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{v}$$

Bernoulli's Trinomial

EQ. OF MOTION for a **barotropic perfect fluid** under potential body forces

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Barotropic perfect fluid with **potential** forces: Steady state solution

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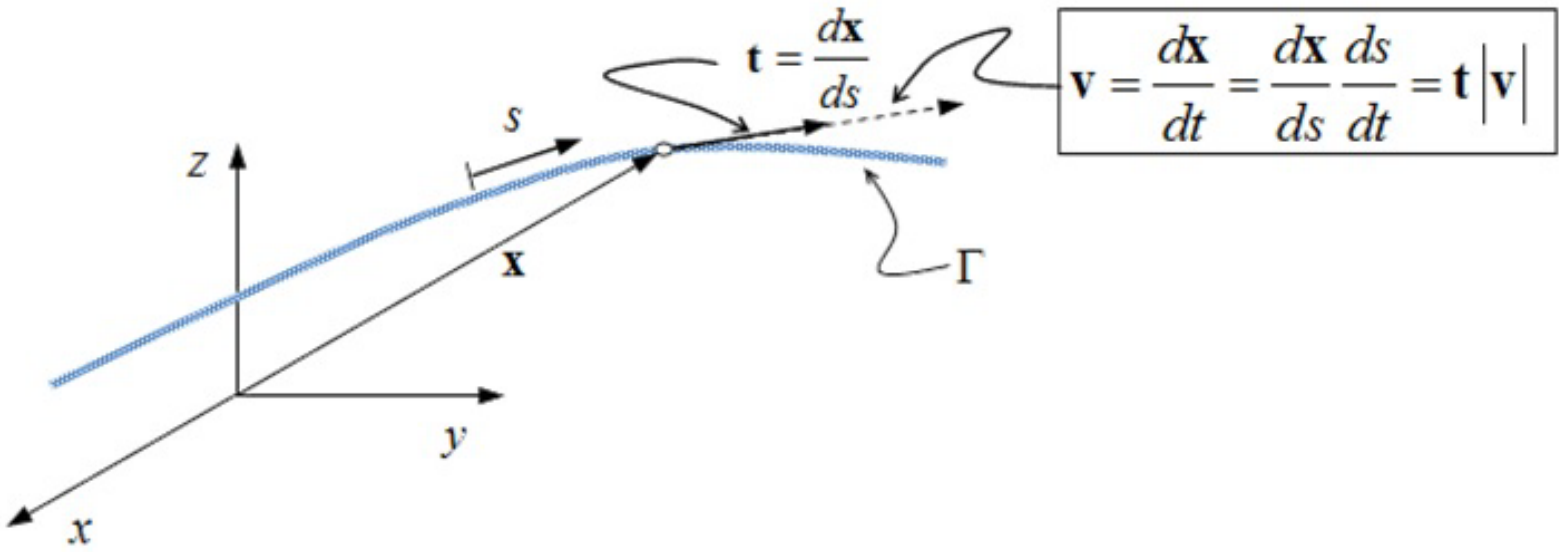
- The **equation of motion** for a **steady flow** becomes:

$$-\nabla\left[\mathbb{P}+\phi+\tfrac{1}{2}\mathrm{v}^2\right]=\cancel{\frac{\partial \mathbf{v}}{\partial t}}+2\boldsymbol{\omega}\times\mathbf{v} \quad \Rightarrow \quad -\nabla\left[\mathbb{P}+\phi+\frac{1}{2}\mathrm{v}^2\right]=2\boldsymbol{\omega}\times\mathbf{v}$$

- Considering **a stream line** $\Gamma:\ \mathbf{x}=\mathbf{x}(s)$ parameterized in terms of its arc length s :

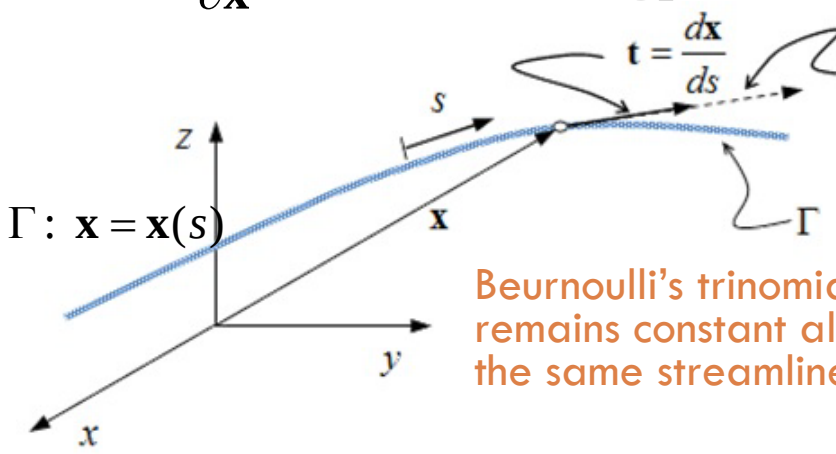


Barotropic perfect fluid with **potential** forces: Steady state solution

- Then, the equation of motion along the considered streamline , Γ, reads:

$$-\nabla\underbrace{\left[\mathbb{P}+\phi+\tfrac{1}{2}\mathrm{v}^2\right]}_{=M(\mathbf{x}(s))}=2\boldsymbol{\omega}\times\mathbf{v} \quad\Rightarrow\quad -(\nabla M)\cdot\underbrace{\mathbf{v}}_{=\frac{d\mathbf{x}}{ds}\frac{ds}{dt}}=\underbrace{(2\boldsymbol{\omega}\times\mathbf{v})\cdot\mathbf{v}}_{=0}$$

$$\underbrace{\nabla M(\mathbf{x}(s))}_{\frac{\partial M}{\partial \mathbf{x}}}\cdot\frac{d\mathbf{x}}{ds}=\frac{dM}{ds}=0 \qquad \forall\mathbf{x}\in\Gamma \quad\Rightarrow\quad \boxed{M(\mathbf{x})=cnt \qquad \forall\mathbf{x}\in\Gamma}$$



$$\mathbf{v}=\frac{d\mathbf{x}}{dt}=\frac{d\mathbf{x}}{ds}\frac{ds}{dt}=\mathbf{t}|\mathbf{v}|$$

Beurnoulli's trinomial remains constant along the same streamline.

$$\left[\mathbb{P}+\phi+\tfrac{1}{2}\mathrm{v}^2\right](\mathbf{x})=cnt \qquad \forall\mathbf{x}\in\Gamma$$

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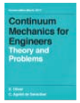
Barotropic perfect fluid with potential (gravitational)forces: steady state solution



- **Incompressible fluid:**

$$\rho = \rho(p) = \rho_0 = cnt \quad \Rightarrow \quad \mathbb{P}(p) \equiv \int_0^p \frac{1}{\underbrace{\rho(\tilde{p})}_{\rho_0}} d\tilde{p} = \frac{1}{\rho_0}\int_0^p d\tilde{p} = \frac{p}{\rho_0}$$

- **Potential gravitational forces:**

$$\mathbf{b}(\mathbf{x}) = -\nabla\phi(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \quad \Rightarrow \quad \phi(\mathbf{x}) = gz$$

$$\left[\mathbb{P} + \phi + \tfrac{1}{2}\mathrm{v}^2\right](\mathbf{x}) = \left[\frac{p}{\rho_0} + gz + \tfrac{1}{2}\mathrm{v}^2\right] = cnt \qquad \forall \mathbf{x} \in \Gamma$$

$$z + \frac{p}{\rho_0 g} + \frac{1}{2}\frac{\mathrm{v}^2}{g} \overset{def}{=} H = cnt \qquad \forall \mathbf{x} \in \Gamma$$

BERNOULLI'S THEOREM

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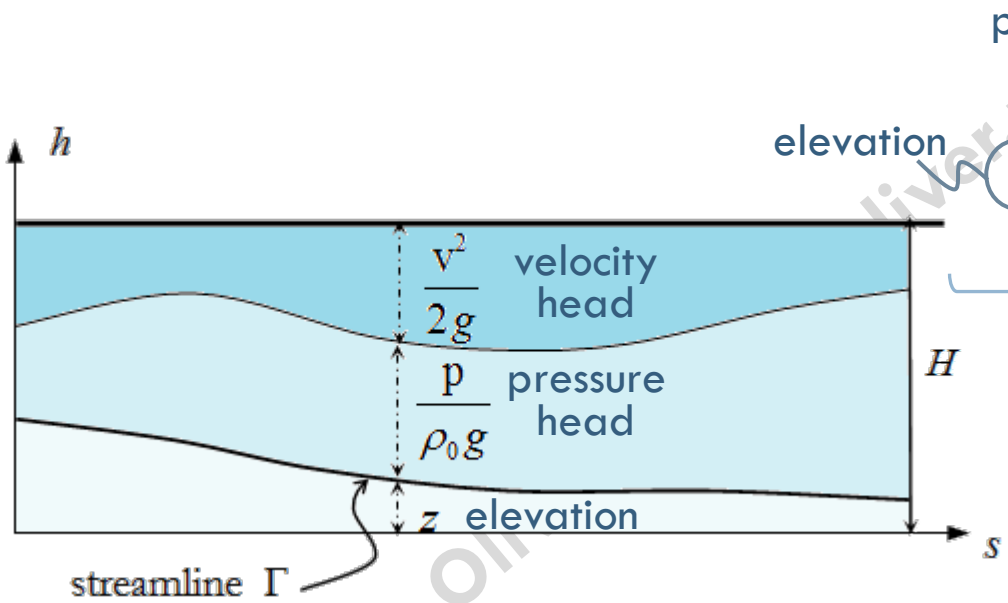
Barotropic perfect fluid with **potential (gravitational)**forces: steady state solution

- Bernoulli's Theorem can be interpreted as:



$$z + \frac{p}{\rho_0 g} + \frac{1}{2}\frac{v^2}{g} \overset{def}{=} H = cnt \qquad \forall \mathbf{x} \in \Gamma$$

elevation; pressure head; velocity head; total or energetic head; Piezometric or hydraulic head, h

- It is a statement of the conservation of mechanical energy:

$$\frac{p}{\rho_0} + gz + \frac{1}{2}v^2 = cnt \qquad \forall \mathbf{x} \in \Gamma$$

pressure energy; potential energy; kinetic energy

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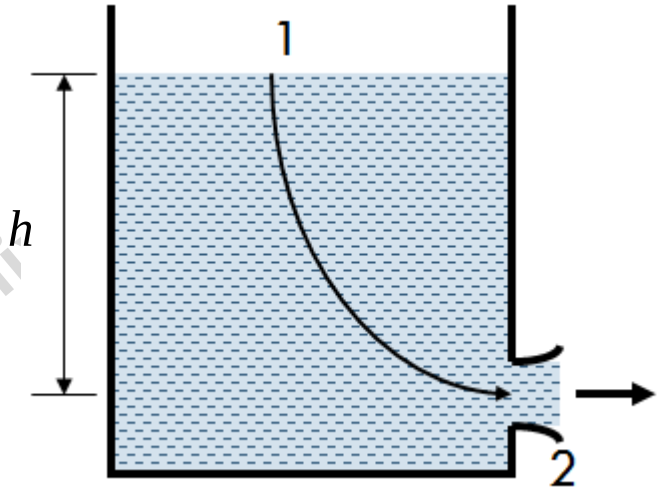
Example

Determine the velocity and mass flow rate of water from the circular hole ($0.1m$ diameter) at the bottom of the water tank (at this instant). The tank is open to the atmosphere and $h = 4m$. Consider a steady state regime.



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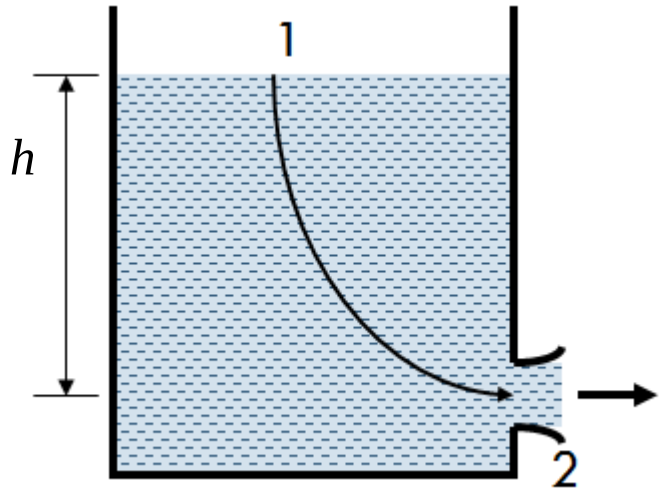
Example - Solution

$S_1 =$ cross section area at 1



$S_2 =$ cross section area at 2

$$z_1 + \frac{p_1}{\rho_0 g} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{\rho_0 g} + \frac{v_2^2}{2g}$$

$$\begin{cases} p_1 = p_2 = p_{atm} \approx 0 \\ v_1 \approx 0 \ (S_1 \geq S_2 \rightarrow v_1 = \underbrace{\frac{S_2}{S_1}}_{\approx 0} v_2 \approx 0) \end{cases}$$

$$z_1 = z_2 + \frac{v_2^2}{2g} \Rightarrow v_2 = \sqrt{2g(z_1 - z_2)}$$

(h)

Velocity at the bottom hole of the tank: $v_2 = \sqrt{2gh}$

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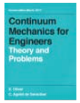
Barotropic perfect fluid with potential forces: Transient solution



Link to YouTube video

- The **equation of motion** for an **unsteady flow** is:

$$-\nabla\left[\mathbb{P}+\phi+\tfrac{1}{2}\mathrm{v}^2\right]=\frac{\partial \mathbf{v}}{\partial t}+2\boldsymbol{\omega}\times\mathbf{v}$$

- This expression may be simplified for:
  - Potential (irrotational) flow
  - Potential and incompressible flow

REMARK

A movement is said to be irrotational (or potential) if the rotational of the velocity field is null at any point:

$$\nabla\times\mathbf{v}(\mathbf{x},t)=\mathbf{0} \qquad \forall\mathbf{x},\ \forall t$$

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Barotropic perfect fluid with potential forces and **irrotational flow**: Transient solution

- In an **irrotational flow**:

$$\nabla \times \mathbf{v}(\mathbf{x},t) = \mathbf{0} \quad \forall \mathbf{x},\ \forall t \quad \Rightarrow \quad \boldsymbol{\omega}(\mathbf{x},t) = \frac{1}{2}\nabla \times \mathbf{v}(\mathbf{x},t) = \mathbf{0} \quad \forall \mathbf{x},\ \forall t$$

  - There will exist a scalar function (named velocity potential $\chi(\mathbf{x},t)$) which satisfies: $\mathbf{v}(\mathbf{x},t) = \nabla\chi(\mathbf{x},t)$

- Then, the **equation of motion** becomes:

$$-\nabla\left[\mathbb{P} + \phi + \tfrac{1}{2}\mathrm{v}^2\right] = \frac{\partial \mathbf{v}}{\partial t} + \cancel{2\boldsymbol{\omega}} \times \mathbf{v} \quad \Rightarrow \quad -\nabla\left[\mathbb{P} + \phi + \tfrac{1}{2}\mathrm{v}^2\right] = \nabla\left(\frac{\partial \chi}{\partial t}\right)$$

$$= \frac{\partial\left(\nabla\chi(\mathbf{x},t)\right)}{\partial t}$$

- Rearranging,

$$-\nabla\left[\mathbb{P} + \phi + \tfrac{1}{2}\mathrm{v}^2 + \frac{\partial \chi}{\partial t}\right] = \mathbf{0} \quad \Rightarrow \quad \nabla M(\mathbf{x},t) = \mathbf{0} \quad \forall \mathbf{x},\ \forall t$$

$$= M(\mathbf{x},t)$$

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Barotropic perfect fluid with potential forces and **irrotational flow**: Transient solution

- The momentum equation can be trivially integrated:

$$\nabla M(\mathbf{x},t)=\mathbf{0} \quad \forall \mathbf{x},\ \forall t \quad \Rightarrow \quad M(\mathbf{x},t)=\mathbb{P}+\phi+\tfrac{1}{2}\mathrm{v}^2+\frac{\partial \chi}{\partial t}=\varphi(t)$$

- Defining a modified velocity potential, $\bar{\chi}(\mathbf{x},t)$:

$$\bar{\chi}(\mathbf{x},t)\overset{def}{=}\chi(\mathbf{x},t)-\int_0^t \varphi(\tau)\,d\tau \quad \Rightarrow \quad \begin{cases} \nabla\bar{\chi}=\nabla\chi=\mathbf{v}(\mathbf{x},t) \\ \dfrac{\partial\bar{\chi}}{\partial t}=\dfrac{\partial\chi}{\partial t}-\varphi(t) \end{cases}$$

- Finally,

$$\boxed{\mathbb{P}+\phi+\frac{1}{2}\left(\nabla\bar{\chi}\right)^2+\frac{\partial\bar{\chi}}{\partial t}=0 \quad \forall \mathbf{x},\ \forall t}$$

Differential equation of **hydraulic transients**

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Barotropic perfect fluid with potential forces: Transient solution in **irrotational** flows

- The **mechanical problem** for a **potential (irrotational) flow:**

$$\mathbf{v}(\mathbf{x},t) = \nabla\overline{\chi}(\mathbf{x},t)$$

$\dot{\rho} + \rho\nabla\cdot\left(\nabla\overline{\chi}(\mathbf{x},t)\right) = 0 \Rightarrow \dot{\rho} + \rho\nabla^2\overline{\chi} = 0$	Conservation of Mass. Continuity Equation.	1 eqn.
$\mathbb{P}(\rho,p) + \phi + \frac{1}{2}\left(\nabla\overline{\chi}\right)^2 + \frac{\partial\overline{\chi}}{\partial t} = 0 \quad \forall\mathbf{x},\ \forall t$	Linear Momentum Balance. Hydraulic Transients Equation.	1 eqn.
$F(\rho,p,\cancel{\theta}) = 0 \Rightarrow \rho = \rho(p)$ barotropic fluid	Kinetic State Equation	1 eqn.

- 3 scalar unknowns: ρ, p, $\overline{\chi}$. Once the potential $\overline{\chi}$ is obtained, the velocity field can be easily calculated:

$$\mathbf{v}(\mathbf{x},t) = \nabla\overline{\chi}(\mathbf{x},t)$$

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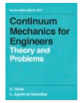
Incompressible perfect fluid with potential forces: transient solution in **irrotational** flows



- In an **incompressible flow:**

$$\frac{d\rho}{dt} = 0 \Rightarrow \rho = \rho_0$$

- Then, the term $\mathbb{P}$ in the equation of motion becomes:

$$\mathbb{P}(\mathbf{x},t) = \int_0^p \frac{1}{\rho(\overline{p})} d\overline{p} = \frac{1}{\rho_0}\int_0^p d\overline{p} = \frac{p}{\rho_0}$$

- And, the **equation of motion** is:

$$-\nabla\left[\frac{p}{\rho_0} + \phi + \frac{1}{2}\left(\nabla\overline{\chi}\right)^2 + \frac{\partial\overline{\chi}}{\partial t}\right] = \mathbf{0}$$

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Fluid Mechanics Equations

- The **mechanical problem** for a **potential (irrotational) and incompressible flow:**

$\cancel{\rho} + \rho \nabla \cdot \left(\nabla \bar{\chi}(\mathbf{x},t) \right) = 0 \Rightarrow \nabla^2 \bar{\chi} \overset{not}{=} \Delta \bar{\chi} = 0$	Conservation of Mass. Continuity Equation.	1 eqn.
$\frac{p}{\rho_0} + \phi + \frac{1}{2} \left(\nabla \bar{\chi} \right)^2 + \frac{\partial \bar{\chi}}{\partial t} = 0 \quad \forall \mathbf{x}, \forall t$	Linear Momentum Balance. Hydraulic Transients Equation.	1 eqn.

- 2 scalar unknowns: p, $\bar{\chi}$. Once the potential $\bar{\chi}$ is obtained, the velocity field can be easily calculated:

$$\mathbf{v}(\mathbf{x},t) = \nabla \bar{\chi}(\mathbf{x},t)$$

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10.4. Newtonian Viscous Fluids

Ch.10. Fluid Mechanics

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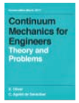

Governing Equations

- **Governing equations** of the general fluid mechanics problem:

$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0$	Conservation of Mass. Continuity Equation.	1 eqn.
$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$	Linear Momentum Balance. Equation of Motion.	3 eqns.
$\rho \dot{u} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \nabla \cdot \mathbf{q}$	Energy Balance. First Law of Thermodynamics.	1 eqn.
$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda Tr(\mathbf{d})\mathbf{1} + 2\mu \mathbf{d}$	Mechanical Constitutive Equations.	6 eqns.
$s = s(\mathbf{d}, \theta, \rho)$	Entropy Constitutive Equation.	1 eqn.
$\mathbf{q} = -K \nabla \theta$	Thermal Constitutive Equation. Fourier's Law of Conduction.	3 eqns.
$u = u(\rho, \theta) \qquad F(\rho, \theta, p) = 0$	Caloric and Kinetic State Equations.	2 eqns.

- 17 scalar unknowns: ρ, $\mathbf{v}$, $\boldsymbol{\sigma}$, u, $\mathbf{q}$, θ, s, p.

Too large to solve analytically !!

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Navier-Stokes Equations

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$

- Consider the following lemmas:
  - ***Lemma 1.*** $$\nabla \cdot \mathbf{d} = \frac{1}{2}\Delta \mathbf{v} + \frac{1}{2}\nabla\left(\nabla \cdot \mathbf{v}\right)$$ Where $\mathbf{d}(\mathbf{x},t)$ is the deformation rate tensor
  - ***Lemma 2.*** $$\nabla \cdot \left(\alpha \mathbf{1}\right) = \nabla \alpha$$ Where $\alpha(\mathbf{x},t)$ is a scalar function.

- Introducing the **constitutive equation** into the **divergence of the stress tensor,** $\nabla \cdot \sigma$ **,**and taking into account these lemmas:

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda \underbrace{Tr\left(\mathbf{d}\right)}_{=\nabla \cdot \mathbf{v}}\mathbf{1} + 2\mu \mathbf{d} \Rightarrow \nabla \cdot \boldsymbol{\sigma} = \nabla \cdot (-p\mathbf{1} + \lambda Tr(\mathbf{d})\mathbf{1} + 2\mu \mathbf{d}) =$$

$$= -\nabla p + \lambda \underbrace{\nabla(Tr(\mathbf{d}))}_{\nabla(\nabla \cdot \mathbf{v})} + \mu \Delta \mathbf{v} + \mu \nabla(\nabla \cdot \mathbf{v})$$

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Navier-Stokes Equations

- Then, the linear **momentum balance equation** is rearranged:

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \quad \Rightarrow \quad -\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{v}) + \mu \Delta \mathbf{v} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt}$$

$$\begin{cases} -\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{v}) + \mu \Delta \mathbf{v} + \rho \mathbf{b} = \rho \dfrac{d\mathbf{v}}{dt} \\ -\dfrac{\partial p}{\partial x_i} + (\lambda + \mu)\dfrac{\partial^2 \mathrm{v}_j}{\partial x_i \partial x_j} + \mu \dfrac{\partial^2 \mathrm{v}_i}{\partial x_j \partial x_j} + \rho b_i = \rho \dfrac{d\mathrm{v}_i}{dt} \qquad i, j \in \{1,2,3\} \end{cases}$$

NAVIER-STOKES EQUATIONS

- The Navier-Stokes equations are essentially the equation of motion (Cauchy's equation) expressed solely in terms of velocity and pressure.

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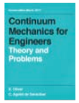

Energy Equation

$$\rho \frac{du}{dt} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \nabla \cdot \mathbf{q}$$

- Consider the definition of **stress power:**

$$\boldsymbol{\sigma} : \mathbf{d} = -pTr(\mathbf{d}) + \kappa Tr^2(\mathbf{d}) + 2\mu \mathbf{d}' : \mathbf{d}'$$

RECOVERABLE POWER, W_R. **DISSIPATIVE POWER,** $2W_D$.

- And the **Fourier's Law:**

$$\mathbf{q} = -K\ \nabla\theta \quad \Rightarrow \quad \nabla \cdot \mathbf{q} = -\nabla \cdot (K\ \nabla\theta)$$

- Introducing these into the **energy balance equation:**

$$\rho \dot{u} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \nabla \cdot \mathbf{q} \quad \Rightarrow$$

$$\rho \frac{du}{dt} = (-pTr(\mathbf{d}) + \kappa Tr^2(\mathbf{d}) + 2\mu \mathbf{d}' : \mathbf{d}') + \rho r + \nabla \cdot (K\ \nabla\theta)$$

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Energy Equation

- Then, the **energy balance equation** is reduced to:

$$\rho\frac{du}{dt}=(-p\underbrace{Tr(\mathbf{d})}_{\nabla\cdot\mathbf{v}}+\kappa Tr^2(\mathbf{d})+2\mu\mathbf{d}':\mathbf{d}')+\rho r+\nabla\cdot(K\nabla\theta)$$

DISSIPATIVE POWER, $2W_D$

$$\begin{cases}\rho\dfrac{du}{dt}=-p\nabla\cdot\mathbf{v}+\rho r+\nabla\cdot(K\nabla\theta)+\kappa\, Tr^2(\mathbf{d})+2\mu\,\mathbf{d}':\mathbf{d}' \\ \rho\dfrac{du}{dt}=-p\dfrac{\partial \mathrm{v_i}}{\partial x_\mathrm{i}}+\rho r+\dfrac{\partial}{\partial x_\mathrm{i}}(K\dfrac{\partial\theta}{\partial x_\mathrm{i}})+\kappa\,(\dfrac{\partial \mathrm{v_i}}{\partial x_\mathrm{i}})^2+2\mu\, d'_{ij}\, d'_{ij} \qquad i,j\in\{1,2,3\}\end{cases}$$

ENERGY EQUATION

- The energy equation is essentially the energy balance equation expressed solely in terms of velocity and pressure.

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Reduced System of Equations

- **Governing equations** of the general fluid mechanics problem are reduced to:

$\dot{\rho}+\rho\nabla\cdot\mathbf{v}=0$	Conservation of Mass. Continuity Equation.	1 eqn.
$-\nabla p+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{v})+\mu\Delta\mathbf{v}+\rho\mathbf{b}=\rho\dot{\mathbf{v}}$	Momentum Balance. Navier-Stokes Equations.	3 eqns.
$\rho\dot{u}=-p\nabla\cdot\mathbf{v}+\rho r+\nabla\cdot(K\nabla\theta)+\kappa\, Tr^2(\mathbf{d})+2\mu\,\mathbf{d}':\mathbf{d}'$	Energy Balance.	1 eqn.
$s=s(\mathbf{v},\theta,\rho)$	Entropy Constitutive Equation.	1 eqn.
$u=u(\rho,\theta)$	Caloric State Equation.	1 eqn.
$F(\rho,\theta,p)=0$	Kinetic State Equation.	1 eqn.

- 8 scalar unknowns: $\rho, \mathbf{v}, p, u, \theta, s$

REMARK

For a barotropic fluid, the mechanic and thermal problems are uncoupled, reducing the mechanical problem to 5 unknowns.

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Navier-Stokes Equations: Physical Interpretation

- The **Navier-Stokes equations** can be physically interpreted as:

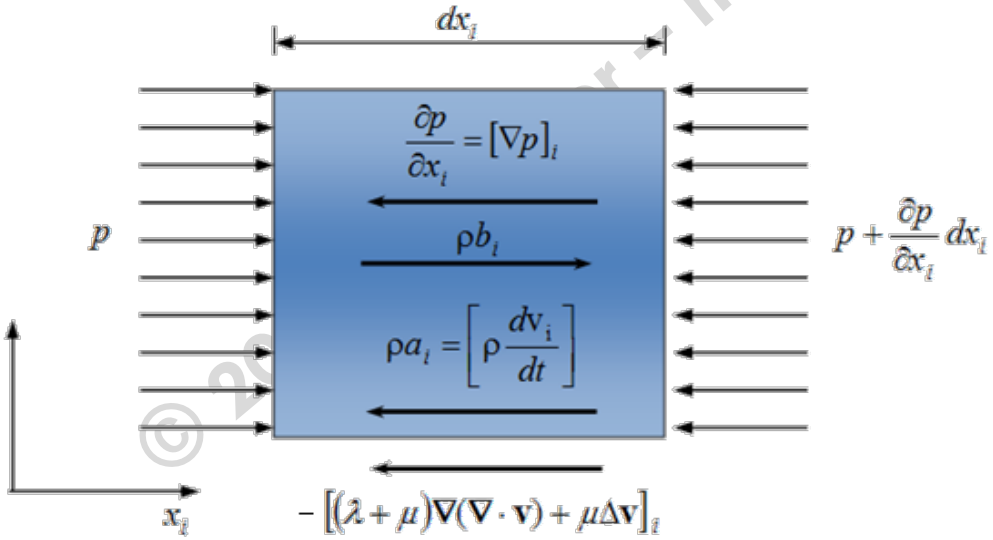
$$-\nabla p - \underbrace{-\left\{\left(\lambda+\mu\right)\nabla\left(\nabla\cdot\mathbf{v}\right)+\mu\Delta\mathbf{v}\right\}}_{>0} + \rho\mathbf{b} - \rho\frac{d\mathbf{v}}{dt} = \mathbf{0}$$

Forces due to the pressure gradient

Viscous forces due to the contact with neighbour particles

Body forces

Inertial forces



NOTE: All forces are per unit of volume.

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Energy Equation: Physical Interpretation

- The **energy equation** can be physically interpreted as:

$$\rho \frac{du}{dt} = -p\nabla\cdot\mathbf{v} + \rho r + \nabla\cdot(K\nabla\theta) + \kappa\, Tr^2(\mathbf{d}) + 2\mu\,\mathbf{d}':\mathbf{d}'$$

Variation of internal energy

Mechanical work of the thermodynamic pressure per unit of time:

Heat generated by the internal sources and conduction per unit volume per unit time

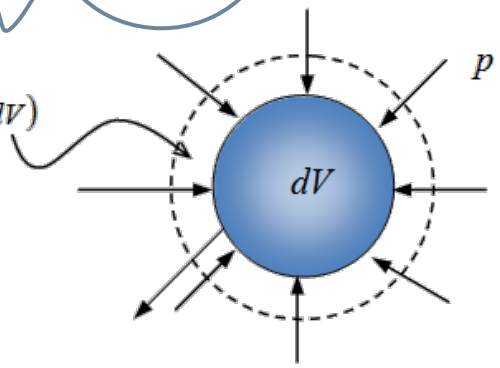
Dissipative power, $2W_D$

$$\frac{d}{dt}(dV) = (\nabla\cdot\mathbf{v})dV$$

$$\nabla\cdot\mathbf{v} = \frac{1}{dV}\frac{d}{dt}(dV)$$

$$-p\nabla\cdot\mathbf{v} = -p\frac{1}{dt}\frac{d}{dV}(dV)$$

Variation of volume per unit of volume and per unit of time



NOTE: All terms are per unit of volume and unit of time

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Fluid Mechanics Equations in Curvilinear Coordinates

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

CONTINUITY EQUATION

Cartesian Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho \mathrm{v}_x) + \frac{\partial}{\partial y}(\rho \mathrm{v}_y) + \frac{\partial}{\partial z}(\rho \mathrm{v}_z) = 0$$

Cylindrical Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r \mathrm{v}_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \mathrm{v}_\theta) + \frac{\partial}{\partial z}(\rho \mathrm{v}_z) = 0$$

Spherical Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 \mathrm{v}_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho \mathrm{v}_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \varphi}(\rho \mathrm{v}_\varphi) = 0$$

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Fluid Mechanics Equations in Curvilinear Coordinates

$$\rho\mathbf{a} = -\nabla p + (\lambda+\mu)\nabla(\cancel{\nabla\cdot\mathbf{v}}) + \mu\Delta\mathbf{v} + \rho\mathbf{b}$$
$$\mathbf{a} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} \quad ; \quad \nabla\cdot\mathbf{v} = \mathbf{0}$$

Link to YouTube video

- **NAVIER-STOKES EQUATIONS** for an **incompressible fluid** with ρ and μ **constants**
  - **Cartesian Coordinates**
    - x component

$$\rho\left(\frac{\partial \mathrm{v}_x}{\partial t} + \mathrm{v}_x\frac{\partial \mathrm{v}_x}{\partial x} + \mathrm{v}_y\frac{\partial \mathrm{v}_x}{\partial y} + v_z\frac{\partial \mathrm{v}_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 \mathrm{v}_x}{\partial x^2} + \frac{\partial^2 \mathrm{v}_x}{\partial y^2} + \frac{\partial^2 \mathrm{v}_x}{\partial z^2}\right) + \rho b_x$$

    - y component

$$\rho\left(\frac{\partial \mathrm{v}_y}{\partial t} + \mathrm{v}_x\frac{\partial \mathrm{v}_y}{\partial x} + \mathrm{v}_y\frac{\partial \mathrm{v}_y}{\partial y} + \mathrm{v}_z\frac{\partial \mathrm{v}_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 \mathrm{v}_y}{\partial x^2} + \frac{\partial^2 \mathrm{v}_y}{\partial y^2} + \frac{\partial^2 \mathrm{v}_y}{\partial z^2}\right) + \rho b_y$$

    - z component

$$\rho\left(\frac{\partial \mathrm{v}_z}{\partial t} + \mathrm{v}_x\frac{\partial \mathrm{v}_z}{\partial x} + \mathrm{v}_y\frac{\partial \mathrm{v}_z}{\partial y} + \mathrm{v}_z\frac{\partial \mathrm{v}_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 \mathrm{v}_z}{\partial x^2} + \frac{\partial^2 \mathrm{v}_z}{\partial y^2} + \frac{\partial^2 \mathrm{v}_z}{\partial z^2}\right) + \rho b_z$$

NOTE: For "slow" motions left-hand-side term is zero ($\rho\mathbf{a} = \mathbf{0}$)

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Fluid Mechanics Equations in Curvilinear Coordinates

$$\rho\mathbf{a} = -\nabla p + (\lambda+\mu)\nabla\left(\cancel{\nabla\cdot\mathbf{v}}\right) + \mu\Delta\mathbf{v} + \rho\mathbf{b}$$
$$\mathbf{a} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} \quad ; \quad \nabla\cdot\mathbf{v} = \mathbf{0}$$

- **NAVIER-STOKES EQUATIONS** for an **incompressible fluid** with ρ and μ **constants**
- **Cylindrical Coordinates**
  - r component

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial^2 v_r}{\partial z^2}\right) + \rho b_r$$

  - θ component

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) + \rho b_\theta$$

  - z component

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho b_z$$

NOTE: For "slow" motions left-hand-side term is zero ($\rho\mathbf{a} = \mathbf{0}$)

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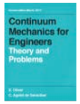
Fluid Mechanics Equations in Curvilinear Coordinates

$$\rho\mathbf{a} = -\nabla p + (\lambda+\mu)\nabla\cancel{(\nabla\cdot\mathbf{v})} + \mu\Delta\mathbf{v} + \rho\mathbf{b}$$
$$\mathbf{a} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} \quad ; \quad \nabla\cdot\mathbf{v} = \mathbf{0}$$



- **NAVIER-STOKES EQUATIONS** for an **incompressible fluid** with ρ and μ **constants**
  - **Spherical Coordinates**
    - r component

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{v_\theta^2+v_\varphi^2}{r}\right) = -\frac{\partial p}{\partial r} +$$
$$+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\varphi^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right) - \frac{2}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}\right) + \rho b_r$$

    - θ component

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi} + \frac{v_r v_\theta}{r} - \frac{v_\varphi^2 cotg\,\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta} +$$
$$+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\varphi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\,cotg\,\theta}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}\right) + \rho b_\theta$$

    - ϕ component

$$\rho\left(\frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_\varphi v_r}{r} + \frac{v_\theta v_\varphi}{r}cotg\,\theta\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi} +$$
$$+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_\varphi\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\varphi}{\partial\varphi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\varphi} + \frac{2\,cotg\,\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\varphi}\right) + \rho b_\varphi$$

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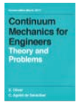
Fluid Mechanics Equations in Curvilinear Coordinates

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda Tr(\mathbf{d})\mathbf{1} + 2\mu\mathbf{d}$$

$$\lambda = \mathrm{K} - \frac{2}{3}\mu \; ; \; Tr(\mathbf{d}) = \nabla \cdot \mathbf{v}$$



- **STRESS TENSOR for Newtonian fluids** $\left(\mathrm{K} = \lambda + \frac{2}{3}\mu\right)$
  - **Cartesian Coordinates**

$$\sigma_x = \mu\left[2\frac{\partial v_x}{\partial x} - \frac{2}{3}(\nabla\cdot\mathbf{v})\right] - p + \mathrm{K}\,\nabla\cdot\mathbf{v} \qquad \tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right]$$

$$\sigma_y = \mu\left[2\frac{\partial v_y}{\partial y} - \frac{2}{3}(\nabla\cdot\mathbf{v})\right] - p + \mathrm{K}\,\nabla\cdot\mathbf{v} \qquad \tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right]$$

$$\sigma_z = \mu\left[2\frac{\partial v_z}{\partial z} - \frac{2}{3}(\nabla\cdot\mathbf{v})\right] - p + \mathrm{K}\,\nabla\cdot\mathbf{v} \qquad \tau_{zx} = \tau_{xz} = \mu\left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right]$$

NOTE: for incompressible fluids $(\nabla\cdot\mathbf{v} = \mathbf{0})$

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Fluid Mechanics Equations in Curvilinear Coordinates

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda Tr(\mathbf{d})\mathbf{1} + 2\mu\mathbf{d}$$

$$\lambda = \mathrm{K} - \frac{2}{3}\mu \; ; \; Tr(\mathbf{d}) = \nabla \cdot \mathbf{v}$$

- **STRESS TENSOR for Newtonian fluids** $\left(\mathrm{K} = \lambda + \frac{2}{3}\mu\right)$
  - **Cylindrical Coordinates**

$$\sigma_r = \mu\left[2\frac{\partial \mathrm{v}_r}{\partial r} - \frac{2}{3}(\nabla\cdot\mathbf{v})\right] - p + \mathrm{K}\ \nabla\cdot\mathbf{v} \qquad \tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\mathrm{v}_\theta}{r}\right) + \frac{1}{r}\frac{\partial \mathrm{v}_r}{\partial \theta}\right]$$

$$\sigma_\theta = \mu\left[2\left(\frac{1}{r}\frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_r}{r}\right) - \frac{2}{3}(\nabla\cdot\mathbf{v})\right] - p + \mathrm{K}\ \nabla\cdot\mathbf{v} \qquad \tau_{\theta z} = \tau_{z\theta} = \mu\left[\frac{\partial \mathrm{v}_\theta}{\partial z} + \frac{1}{r}\frac{\partial \mathrm{v}_z}{\partial \theta}\right]$$

$$\sigma_z = \mu\left[2\frac{\partial \mathrm{v}_z}{\partial z} - \frac{2}{3}(\nabla\cdot\mathbf{v})\right] - p + \mathrm{K}\ \nabla\cdot\mathbf{v} \qquad \tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial \mathrm{v}_z}{\partial r} + \frac{\partial \mathrm{v}_r}{\partial z}\right]$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\mathrm{v}_r) + \frac{1}{r}\frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\partial \mathrm{v}_z}{\partial z}$$

NOTE: for incompressible fluids $(\nabla\cdot\mathbf{v} = \mathbf{0})$

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Fluid Mechanics Equations in Curvilinear Coordinates

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda Tr(\mathbf{d})\mathbf{1} + 2\mu\mathbf{d}$$

$$\lambda = \mathrm{K} - \frac{2}{3}\mu \; ; \; Tr(\mathbf{d}) = \nabla \cdot \mathbf{v}$$

- **STRESS TENSOR for Newtonian fluids** $\left(\mathrm{K} = \lambda + \frac{2}{3}\mu\right)$
  - **Spherical Coordinates**

$$\sigma_r = \mu\left[2\frac{\partial \mathrm{v}_r}{\partial r} - \frac{2}{3}(\nabla\cdot\mathbf{v})\right] - p + \mathrm{K}\,\nabla\cdot\mathbf{v}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\mathrm{v}_\theta}{r}\right) + \frac{1}{r}\frac{\partial \mathrm{v}_r}{\partial \theta}\right]$$

$$\sigma_\theta = \mu\left[2\left(\frac{1}{r}\frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_r}{r}\right) - \frac{2}{3}(\nabla\cdot\mathbf{v})\right] - p + \mathrm{K}\,\nabla\cdot\mathbf{v}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\mathrm{v}_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_\theta}{\partial\phi}\right]$$

$$\sigma_\varphi = \mu\left[2\left(\frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_\varphi}{\partial\varphi} + \frac{\mathrm{v}_r}{r} + \frac{\mathrm{v}_\theta cotg\,\theta}{r}\right) - \frac{2}{3}(\nabla\cdot\mathbf{v})\right] - p + \mathrm{K}\,\nabla\cdot\mathbf{v}$$

$$\tau_{\phi r} = \tau_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{\mathrm{v}_\phi}{r}\right)\right]$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mathrm{v}_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\mathrm{v}_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_\varphi}{\partial\varphi}$$

NOTE: for incompressible fluids $(\nabla\cdot\mathbf{v} = \mathbf{0})$

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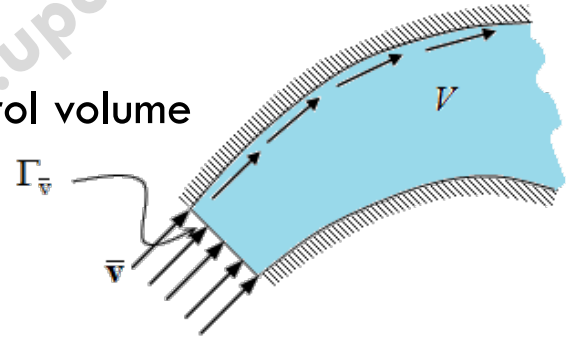
10.5. Boundary Conditions

Ch.10. Fluid Mechanics

Boundary Conditions in Velocities

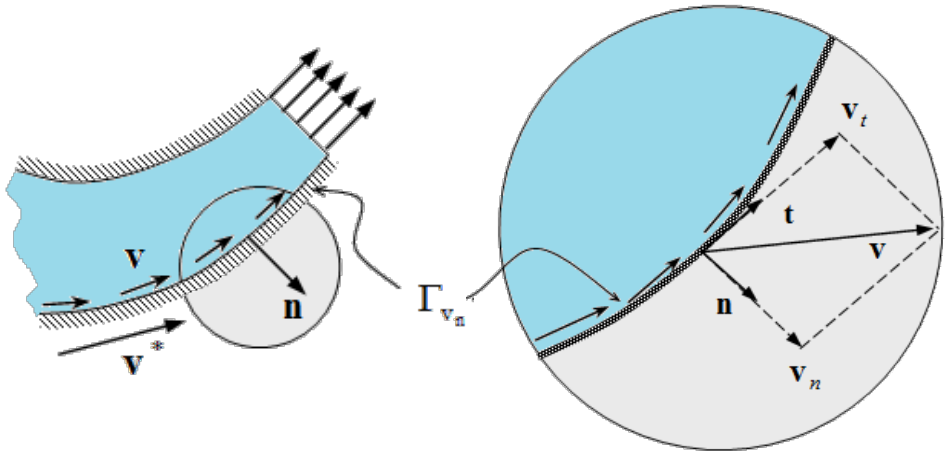
- Prescribed velocities
  - Velocities are known in a certain part of the control volume boundary, Γ_{v} : $\mathbf{v}(\mathbf{x},t)=\overline{\mathbf{v}}(\mathbf{x},t) \quad \forall \mathbf{x}\in\Gamma_{\mathrm{v}}$



- Impervious walls
  - Part of the boundary of control volume, Γ_{v_n} , which can be mobile, is **impervious** (it cannot be penetrated by the fluid).
  - The **normal component** of the **relative** fluid/wall velocity, $\mathbf{v}_r \equiv \mathbf{v}-\mathbf{v}^*$, is considered null.

$$\left(\mathbf{v}-\mathbf{v}^*\right)\cdot\mathbf{n}=0 \quad \forall \mathbf{x}\in\Gamma_{\mathrm{v}_n}$$

$$\mathrm{v}_{\mathrm{n}}(\mathbf{x},t)=\mathbf{v}\cdot\mathbf{n}=\mathbf{v}^*\cdot\mathbf{n} \quad \forall \mathbf{x}\in\Gamma_{\mathrm{v}_n}$$



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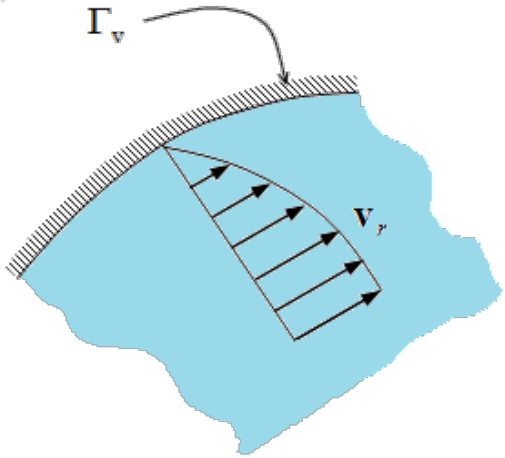
Boundary Conditions in Velocities

- Adherent walls
  - In a **viscous fluid** in contact with a wall the fluid is considered to adhere to the wall.
  - The relative fluid/wall velocity, $\mathbf{v}_r$, is considered null.

$$\mathbf{v}_r(\mathbf{x},t) = \mathbf{v} - \mathbf{v}^* = \mathbf{0} \quad \forall \mathbf{x} \in \Gamma_v$$

$$\mathbf{v} = \mathbf{v}^* \quad \forall \mathbf{x} \in \Gamma_v$$



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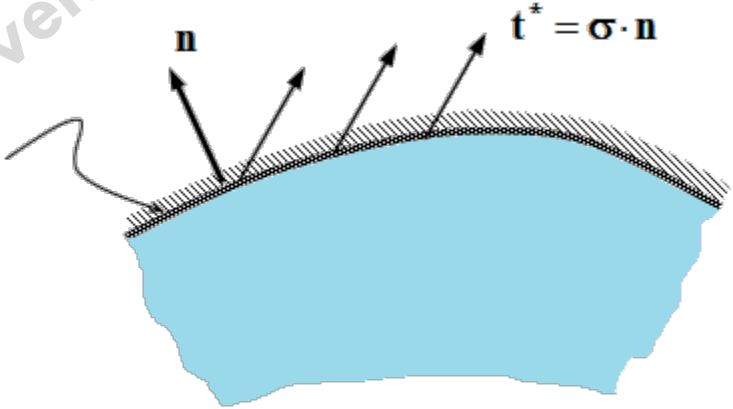
Link to YouTube video

Boundary Conditions in Pressures

- Prescribed tractions
  - The traction vector's value is prescribed in certain parts of the control volume contour Γ_σ:

$$\mathbf{t}(\mathbf{x},t) = \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{t}^*(\mathbf{x},t) \quad \forall \mathbf{x} \in \Gamma_\sigma$$



  - Sometimes, only part of the traction vector is prescribed, such as the thermodynamic pressure.
  - For a Newtonian fluid:

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda Tr(\mathbf{d})\mathbf{1} + 2\mu\mathbf{d} \quad \Rightarrow \quad \mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n} = -p\mathbf{n} + \lambda Tr(\mathbf{d})\mathbf{n} + 2\mu\mathbf{d}\cdot\mathbf{n}$$

$$p(\mathbf{x},t) = p^*(\mathbf{x},t) \quad \forall \mathbf{x} \in \Gamma_p$$

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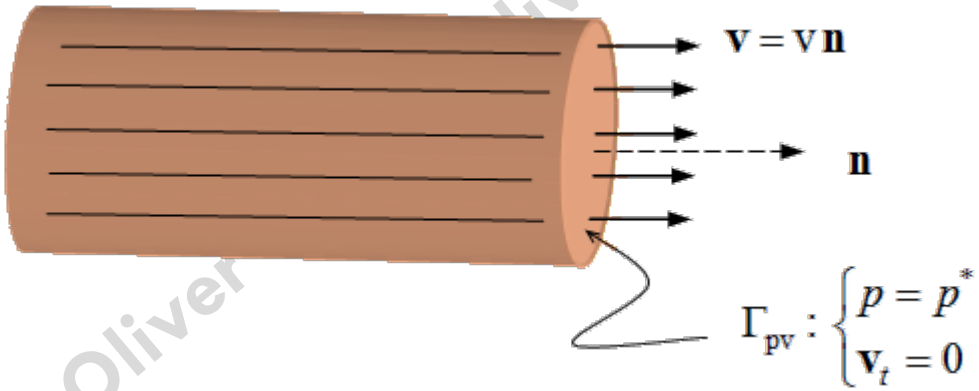
Mixed Boundary Conditions

- Prescribed traction vector and velocities
  - Pressure and the tangential component of the velocity, $\mathbf{v}_t$, are prescribed :



  - This boundary condition is typically used in problems involving in-flow and out-flow sections (pipes).

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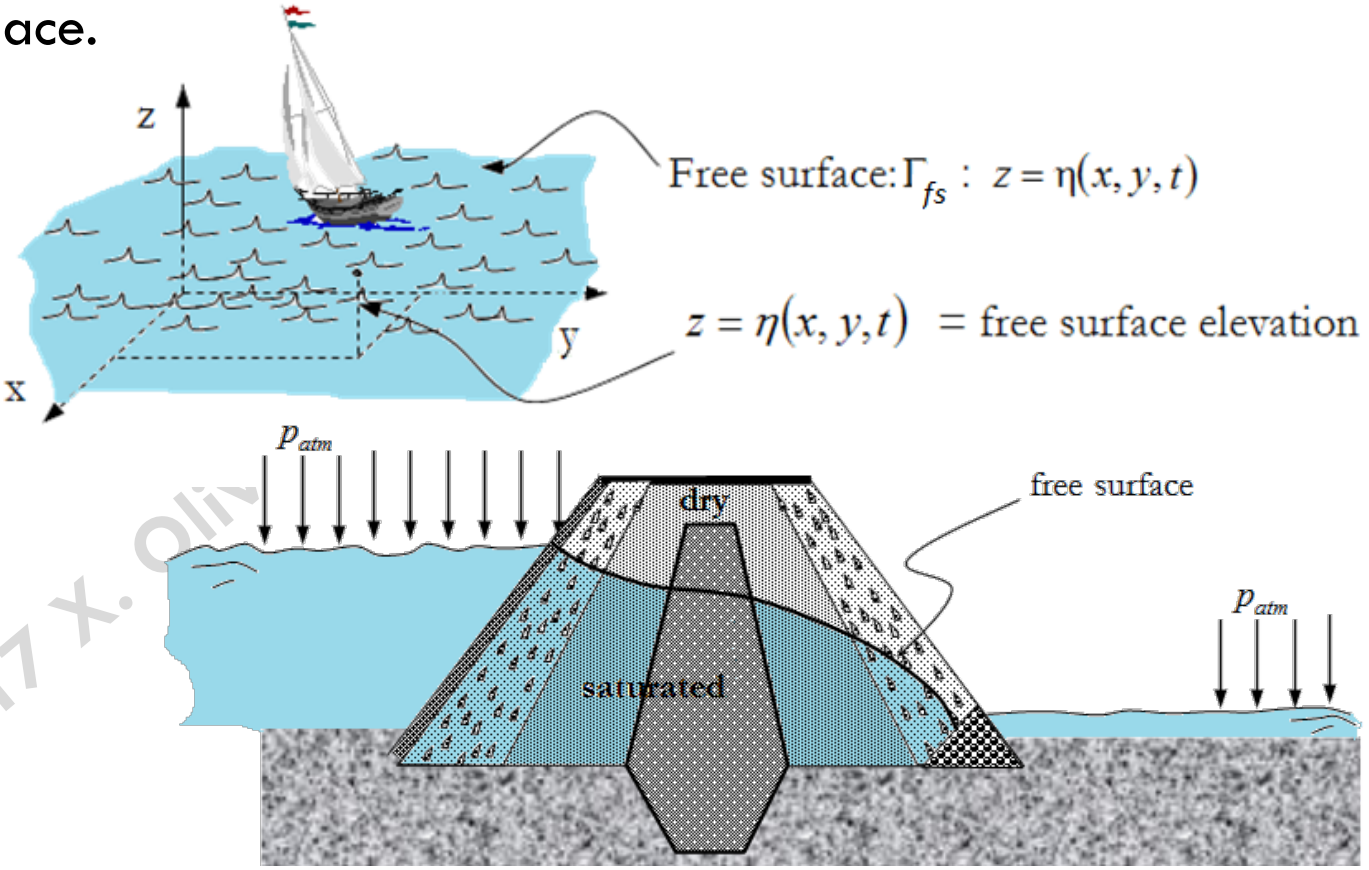
Boundary Conditions on Free Surfaces

- The contact surface between air and fluid (generally water) is a free surface.



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Boundary Conditions on Free Surfaces

- HYPOTHESIS: The free surface is a material surface.
  - This implicitly establishes certain boundary conditions on the velocity field of the material surface Γ_{fs}.
  - Consider the free surface:

$$\Gamma_{fs} := \{\mathbf{x} \mid \phi(x,y,z,t) \equiv z - \eta(x,y,t) = 0\}$$

  - Impose the condition for a material surface (null material derivative):

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \mathbf{v}\cdot\nabla\phi = -\frac{\partial \eta}{\partial t} - v_x \frac{\partial \eta}{\partial x} - v_y \frac{\partial \eta}{\partial y} + v_z \underbrace{\frac{\partial \phi}{\partial z}}_{=1} = 0$$

$$\Downarrow$$

$$v_z(\mathbf{x},t) = \frac{\partial \eta}{\partial t} + v_x \frac{\partial \eta}{\partial x} + v_y \frac{\partial \eta}{\partial y} \quad \forall \mathbf{x} \in \Gamma_{fs}$$

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Boundary Conditions on Free Surfaces

- Another boundary condition typically used on free surfaces is:

$$p\left(\mathbf{x},t\right)=P_{atm}\qquad\forall\mathbf{x}\in\Gamma_{fs}$$

- This allows identifying the position of the free surface once the pressure field is known:

$$\Gamma_{fs}:=\{\mathbf{x}\ \mid\ p\left(\mathbf{x},t\right)-P_{atm}=0\ \}$$

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10.6. Laminar and Turbulent Flows

Ch.10. Fluid Mechanics

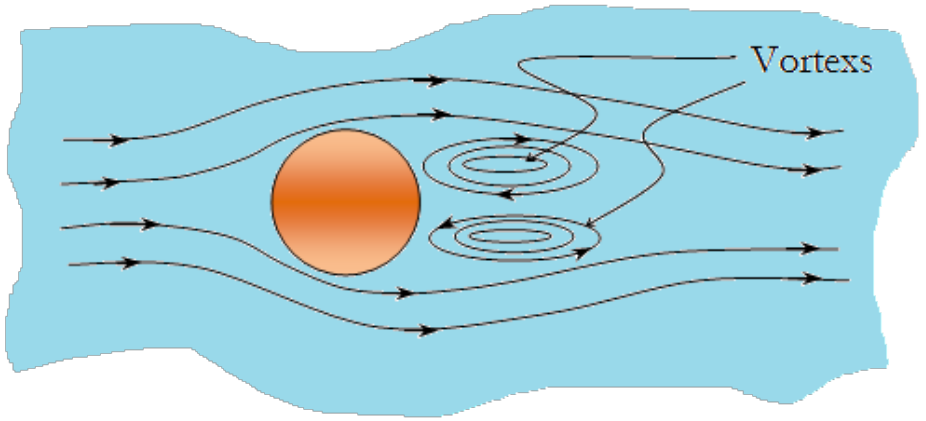
Link to YouTube video

Laminar Flow

- Flow persists as unidirectional movement.
  - Particles flow in parallel layers which do not mix.
  - A flow's laminar character is identified by the Reynolds number: $R_e < 1000$
  - The governing equations of the fluid mechanics problem are valid for this type of flow.

$$R_e \overset{def}{=} \frac{V \times L}{\nu}$$

V, Flow's characteristic velocity
L, Domain's characteristic length
ν, Kinematic viscosity: $\nu = \mu / \rho$



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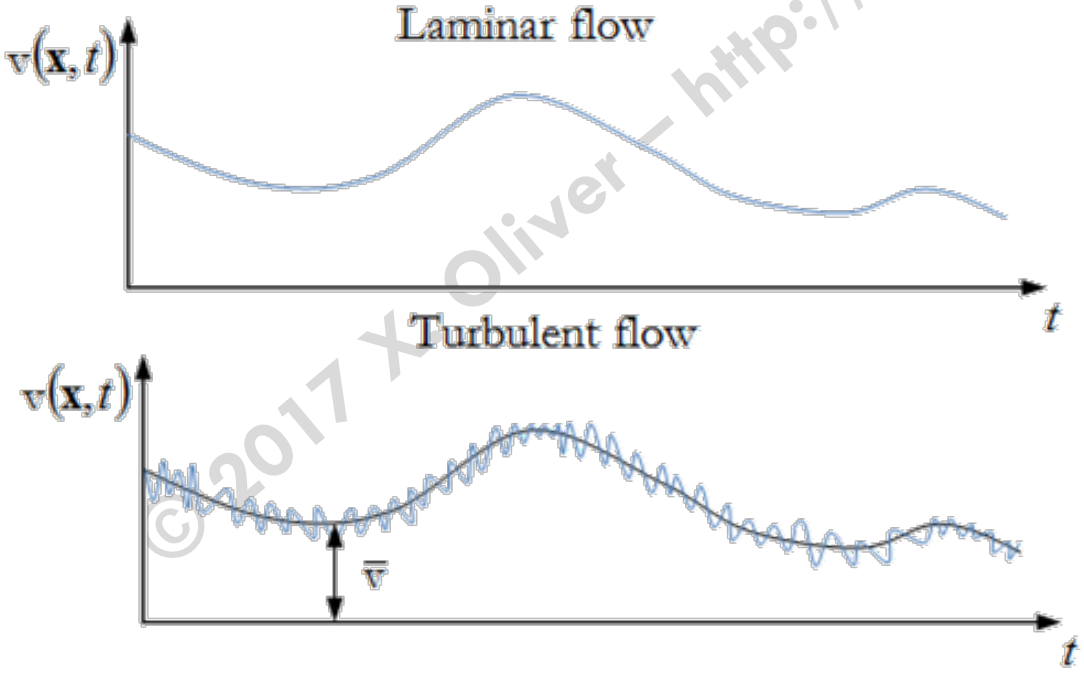
Turbulent Flow

- High values of the Reynolds number.
- Highly distorted and unstable flow:
  - Stress and velocity at a **given spatial point** fluctuate randomly and very fast, along time, about a mean value.
  - Specific models (turbulence models) are used to characterize this regime.



NOTE: Turbulent flow is out of the scope of this course

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Chapter 10
Fluid Mechanics

10.1 Governing Equations

A fluid is a particular case of continuous medium that is characterized by its specific set of constitutive equations. Consequently, the fluid mechanics problem is defined by the following equations:

a) **Balance Equations**

1) *Mass continuity equation*

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \qquad \text{(1 equation)} \tag{10.1}$$

2) *Balance of linear momentum*

$$\nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \qquad \text{(3 equations)} \tag{10.2}$$

3) *Energy balance*

$$\rho \frac{du}{dt} = \boldsymbol{\sigma} : \mathbf{d} + \rho\, r - \nabla \cdot \mathbf{q} \qquad \text{(1 equation)} \tag{10.3}$$

4) *Restrictions imposed by the second law of thermodynamics*

$$\begin{array}{ll} \text{Clausius-Planck inequality} & -\rho \left(\dfrac{du}{dt} - \theta \dfrac{ds}{dt} \right) + \boldsymbol{\sigma} : \mathbf{d} \geq 0 \\ \text{Heat conduction inequality} & -\dfrac{1}{\rho \theta^2}\, \mathbf{q} \cdot \nabla \theta \geq 0 \end{array} \tag{10.4}$$

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b) **Constitutive Equations**

5) *Thermo-mechanical constitutive equation*

$$\boldsymbol{\sigma} = -p\,\mathbf{1} + \lambda\,\mathrm{Tr}\,(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d} \qquad \text{(6 equations)} \tag{10.5}$$

6) *Entropy constitutive equation*

$$s = s\,(\mathbf{d}, \rho, \theta) \qquad \text{(1 equation)} \tag{10.6}$$

7) *Law of heat conduction*

$$\mathbf{q} = -K\,\nabla\theta \qquad \text{(1 equation)} \tag{10.7}$$

c) **Thermodynamic equations of state**

8) *Caloric equation of state*

$$u = u\,(\rho, \theta) \qquad \text{(1 equation)} \tag{10.8}$$

9) *Kinetic equation of state*

$$F\,(\rho, p, \theta) = 0 \qquad \text{(1 equation)} \tag{10.9}$$

The unknowns[1] of these governing equations are

$$\left.\begin{array}{ll} \rho & \rightarrow \text{ 1 unknown} \\ \mathbf{v} & \rightarrow \text{ 3 unknowns} \\ \boldsymbol{\sigma} & \rightarrow \text{ 6 unknowns} \\ u & \rightarrow \text{ 1 unknown} \\ \mathbf{q} & \rightarrow \text{ 3 unknowns} \\ \theta & \rightarrow \text{ 1 unknown} \\ s & \rightarrow \text{ 1 unknown} \\ p & \rightarrow \text{ 1 unknown} \end{array}\right\} \rightarrow \text{ 17 unknowns .} \tag{10.10}$$

The system is formed by a total of 17 PDEs and 17 unknowns which, in general, should be solved together, that is, in a coupled form. However, as noted in Section 5.13.1 of Chapter 5, under certain hypotheses or situations a reduced system

[1] Note that the strain rate tensor **d** is not considered an unknown since it is an implicit function of the velocity field **v**.

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of equations, denoted as the mechanical problem, may be posed and solved separately for a reduced number of unknowns (mechanical variables).

Consider the case of a *barotropic fluid*, which is characterized by the fact that the temperature does not intervene in the kinetic equation of state. Then,

$$\underset{\text{of state}}{\text{Kinetic equation}} \qquad F(\rho, p) = 0 \quad \Longrightarrow \quad \rho = \rho(p) \ , \tag{10.11}$$

which establishes that the density may be described solely by means of the thermodynamic pressure (see Figure 10.1). Assuming, in addition, that the temperature does not intervene in the thermo-mechanical constitutive equation (10.5), the *governing equations of the (uncoupled) mechanical problem in a Newtonian fluid* are defined as

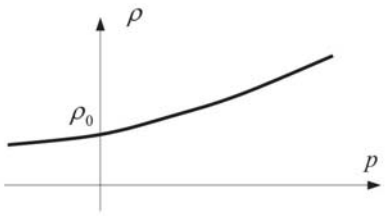

Figure 10.1: Density depends on the thermodynamic pressure in a barotropic fluid.

1) *Mass continuity equation*

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \qquad \text{(1 equation)} \tag{10.12}$$

2) *Cauchy's equation*

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \qquad \text{(3 equations)} \tag{10.13}$$

3) *Mechanical constitutive equation*

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda \, \mathrm{Tr}(\mathbf{d})\mathbf{1} + 2\mu \, \mathbf{d} \qquad \text{(6 equations)} \tag{10.14}$$

4) *Kinetic equation of state*

$$\rho = \rho(p) \qquad \text{(1 equation)} \tag{10.15}$$

The unknowns of the problem posed by the equations above are

$$\left.\begin{array}{ll} \rho & \rightarrow \ 1 \text{ unknown} \\ \mathbf{v} & \rightarrow \ 3 \text{ unknowns} \\ \boldsymbol{\sigma} & \rightarrow \ 6 \text{ unknowns} \\ p & \rightarrow \ 1 \text{ unknown} \end{array}\right\} \ \rightarrow \ 11 \text{ unknowns} \ . \tag{10.16}$$

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A reduced system of 11 equations and 11 unknowns (mechanical problem) is obtained, which may be solved uncoupled from the rest of the problem (thermal problem).

10.2 Hydrostatics. Fluids at Rest

Consider the following particular cases in terms of a fluid's velocity:

a) Uniform velocity: $\mathbf{v}(\mathbf{x},t) \equiv \mathbf{v}(t)$

In this case, the spatial description of the velocity does not depend on the spatial point being considered and is only a function of time. Therefore,

$$\mathbf{d} = \nabla^S \mathbf{v} = \frac{1}{2}(\mathbf{v}\otimes\nabla + \nabla\otimes\mathbf{v}) = \mathbf{0}\,. \tag{10.17}$$

Then, the constitutive equation (10.14) is reduced to

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda \operatorname{Tr}\underbrace{(\mathbf{d})}_{=\mathbf{0}}\mathbf{1} + 2\mu\underbrace{\mathbf{d}}_{=\mathbf{0}} \quad \Longrightarrow \quad \boxed{\boldsymbol{\sigma} = -p\mathbf{1}}\,, \tag{10.18}$$

which indicates that the stress state is hydrostatic (see Figure 10.2). In addition, the mean pressure $\bar{p}$ and the thermodynamic pressure p coincide,

$$\operatorname{Tr}(\boldsymbol{\sigma}) = -3\bar{p} = -3p \quad \Longrightarrow \quad \boxed{\bar{p} = p}\,. \tag{10.19}$$

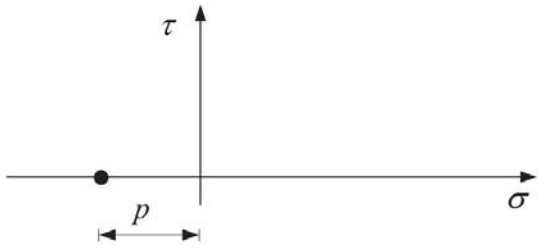

Figure 10.2: Mohr's circle for a fluid with uniform velocity.

b) Uniform and stationary velocity: $\mathbf{v}(\mathbf{x},t) \equiv const$.

A fluid with uniform and stationary velocity is characterized, in addition of (10.17), by

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$$\left.\begin{aligned} &\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{0} \\ &\boldsymbol{\sigma} = -p_0\mathbf{1} \quad \Longrightarrow \quad \bar{p} = p = p_0 \end{aligned}\right\} \text{ hydrostatic case} \tag{10.20}$$

This is the most general case of *hydrostatics*, which is characterized by a null acceleration (the velocity of each particle is constant, although not necessarily null) and the three pressures (thermodynamic p, mean $\bar{p}$, and hydrostatic p_0) coincide.

c) Fluid at rest: $\mathbf{v}(\mathbf{x},t) \equiv const. = \mathbf{0}$

A particular case of hydrostatics is that of a fluid at rest with null velocity.

10.2.1 Hydrostatic Equations

The hydrostatic problem is governed by the following equations:

1) *Constitutive equation*

$$\begin{aligned} &\boldsymbol{\sigma} = -p_0\mathbf{1} \\ &\sigma_{ij} = -p_0\,\delta_{ij} \qquad i,j \in \{1,2,3\} \end{aligned} \tag{10.21}$$

where p_0 is the hydrostatic pressure.

Remark 10.1 *Pascal's Principle* states that, in a fluid at rest, the pressure is the same in every direction.
This classic fluid mechanics postulate is guaranteed by the spherical structure of the stress tensor in (10.21), which ensures that all directions are principal stress directions (see Figure 10.3).

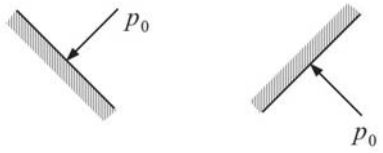

Figure 10.3: Pascal's Principle.

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2) *Mass continuity equation*

$$\left.\begin{array}{l} \dfrac{d\rho}{dt}+\rho\nabla\cdot\mathbf{v}=\mathbf{0} \\ \mathbf{v}=const. \;\Rightarrow\; \nabla\cdot\mathbf{v}=0 \end{array}\right\} \Rightarrow \;\frac{d\rho}{dt}=0 \;\Rightarrow\; \boxed{\rho(\mathbf{X},t)=\rho_0(\mathbf{X})=const.} \tag{10.22}$$

and the density of a same particle does not change along time.

3) *Cauchy's equation*

$$\nabla\cdot\boldsymbol{\sigma}+\rho\,\mathbf{b}=\rho\frac{d\mathbf{v}}{dt} \tag{10.23}$$

Introducing (10.21) and (10.22) in (10.23),

$$\begin{cases} \nabla\cdot\boldsymbol{\sigma}=\nabla\cdot(-p_0\,\mathbf{1})=-\nabla p_0 \\ \left[\nabla\cdot\boldsymbol{\sigma}\right]_j=\dfrac{\partial\sigma_{ij}}{\partial x_i}=\dfrac{\partial}{\partial x_i}(-p_0\,\delta_{ij})=-\dfrac{\partial p_0}{\partial x_j}=-\left[\nabla p_0\right]_j \qquad j\in\{1,2,3\} \end{cases} \tag{10.24}$$

$$\text{Fundamental equation of hydrostatics} \quad \begin{cases} -\nabla p_0+\rho_0\,\mathbf{b}=\mathbf{0} \\ -\dfrac{\partial p_0}{\partial x_i}+\rho_0\,b_i=0 \qquad i\in\{1,2,3\} \end{cases} \tag{10.25}$$

10.2.2 Gravitational Force. Triangular Pressure Distribution

Consider the particular case, which is in fact very common, of the body forces $\mathbf{b}(\mathbf{x},t)$ corresponding to the gravitational force (assumed constant in space and along time, and oriented in the negative direction of the x_3-axis, as shown in Figure 10.4).

Since the acceleration is null (see (10.20)) it is a quasi-static problem and, because the actions $\mathbf{b}(\mathbf{x},t)\equiv const.$ are independent of time, so are the responses, in particular, the hydrostatic pressure. Consequently,

$$p_0(\mathbf{x},t)\equiv p_0(\mathbf{x})=p_0(x,y,z)\;, \tag{10.26}$$

and (10.25) can be integrated as follows

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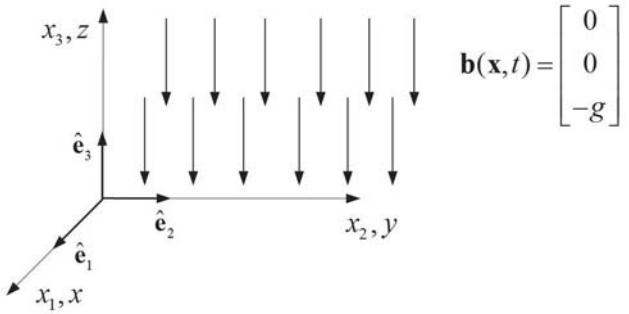

Figure 10.4: Gravitational Force.

$$\begin{cases} -\dfrac{\partial p_0(x,y,z)}{\partial x} = 0 & \Longrightarrow \quad p_0(x,y,z) \equiv p_0(y,z) \\ -\dfrac{\partial p_0(y,z)}{\partial y} = 0 & \Longrightarrow \quad p_0(y,z) \equiv p_0(z) \\ -\dfrac{\partial p_0(z)}{\partial z} - \rho_0 g = 0 & \Longrightarrow \quad p_0 = -\rho_0 g z + C \end{cases} \tag{10.27}$$

For a case such as the one shown in Figure 10.5, in which the surface pressure (height $z = h$) is considered null, the solution (10.26) results in

$$p_0\Big|_{z=h} = 0 \;\Rightarrow\; -\rho_0 g h + C = 0 \;\Rightarrow\; C = \rho_0 g h \;\Rightarrow\; \boxed{p_0 = \rho_0 g\,(h - z)}\,, \tag{10.28}$$

which corresponds to a triangular pressure distribution, as shown in Figure 10.5.

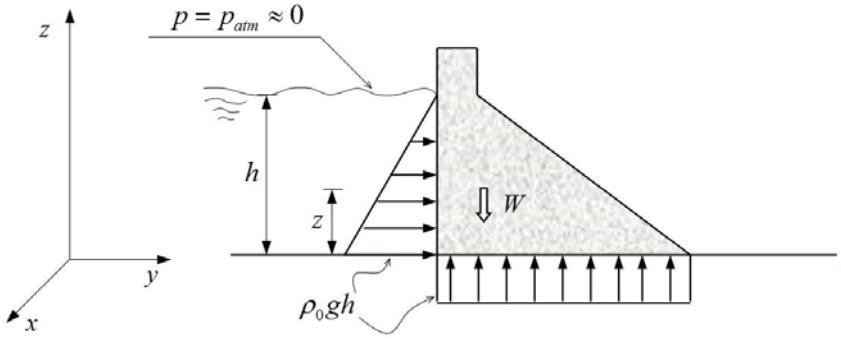

Figure 10.5: Pressure distribution on a gravity dam.

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10.2.3 Archimedes' Principle

Definition 10.1. Archimedes' principle:

1) The upward buoyant force experienced by a body submerged in a fluid is equal to the weight of the fluid displaced by said body.

The classical principle is complemented with:

2) The resultant of the aforementioned buoyant force acts at the center of gravity of the volume of the displaced fluid.

To prove Archimedes' principle, consider the situations in Figure 10.6. On the one hand, Figure 10.6 *a)* illustrates a solid with volume V and density ρ in the interior of a fluid of density ρ_0. The solid *is not necessarily in equilibrium*, even though its velocity and acceleration are assumed to be small enough to ensure a hydrostatic state in the fluid. On the other hand, Figure 10.6 *b)* shows the same fluid without the solid, such that the volume occupied by said solid in Figure 10.6 *a)* is occupied here by an identical volume of fluid.

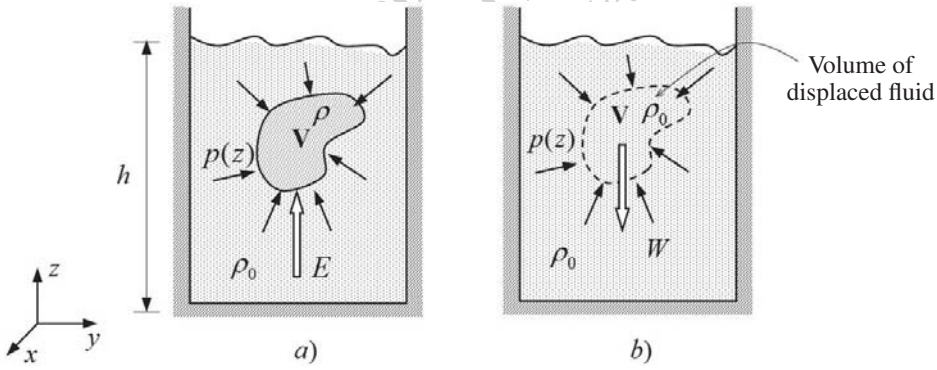

Figure 10.6: *a)* Solid submerged in a fluid and *b)* volume of the displaced fluid.

a) Pressure and stress distributions in the fluid

Using the fundamental equation of hydrostatics (10.25) and considering that the gravitational forces act in the negative direction of the z-axis, the situation corresponding with (10.26) and (10.27) is achieved. Thus, the result (10.28) is valid for both cases *a)* and *b)* of Figure 10.6.

$$\boxed{\begin{aligned} p_0(z) &= \rho_0 g (h - z) \\ \boldsymbol{\sigma} &= -p_0 \mathbf{1} \end{aligned}} \tag{10.29}$$

Note that the hydrostatic pressure and the stress state in the fluid are the same for equivalent points of the fluid in the cases *a)* and *b)* of Figure 10.6.

b) Buoyant force on the submerged solid

The traction vector on the boundary of the submerged solid in Figure 10.6 *a)* is

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} = -p_0 \mathbf{1} \cdot \mathbf{n} = -p_0 \mathbf{n} \tag{10.30}$$

and the resultant $\mathbf{R}$ of the forces the fluid exerts on the solid is

$$\mathbf{R} = \int_{\partial V} \mathbf{t}\, dS = \int_{\partial V} -p_0 \mathbf{n}\, dS \,. \tag{10.31}$$

Note now that, since the hydrostatic pressure distribution is the same in both cases of Figure 10.6, this resultant is the same as the one obtained in case *b)* for the forces that the rest of the fluid exerts *on the volume of displaced fluid*, with the particularity that, because the pressure distribution is constant in space (with value p_0), the Divergence Theorem (Stokes' Theorem) can be applied on (10.30), resulting in

$$\mathbf{R} = \int_{\partial V} -p_0 \mathbf{n}\, dS = \int_V -\nabla p_0\, dV \,. \tag{10.32}$$

Introducing (10.25) in (10.32) yields

$$\mathbf{R} = \int_V -\nabla p_0\, dV = \int_V -p_0 \mathbf{b}\, dV = \underbrace{-\int_V \rho_0 \mathbf{b}\, dV}_{W \hat{\mathbf{e}}_z} = W \hat{\mathbf{e}}_z = E \hat{\mathbf{e}}_z \,, \tag{10.33}$$

where E is the upward buoyant force acting on the submerged solid and W is the weight of the displaced fluid (see Figure 10.6 *b)*). That is,

$$\underbrace{\text{upward buoyant force}}_{E} = \underbrace{\text{weight of the displaced fluid}}_{W} \,, \tag{10.34}$$

whereby the first part of Archimedes' Principle is proven.

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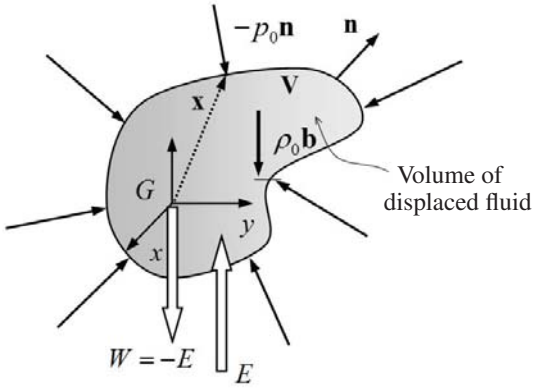

Figure 10.7: Forces acting on the volume of displaced fluid.

c) Vertical line of application of the upward buoyant force

Consider now the moment $\mathbf{M}_E^G$ of the upward buoyant force E with respect to the center of gravity, G, of the volume of displaced fluid (see Figure 10.7[2]),

$$\left\{\begin{array}{l} \mathbf{M}_E^G = \displaystyle\int_{\partial V} \mathbf{x}\times(-p_0\,\mathbf{n})\,dS \overset{\text{Divergence Theorem}}{=} \int_V \mathbf{x}\times(-p_0\nabla)\,dV = -\int_V \mathbf{x}\times\nabla p_0\,dV \\ \left[\mathbf{M}_E^G\right]_i = -\displaystyle\int_{\partial V} e_{ijk}x_j\,p_0\,n_k\,dS = -\int_V \frac{\partial}{\partial x_k}\left(e_{ijk}x_j\,p_0\right)dV = \\ \quad = -\displaystyle\int_V \underbrace{e_{ijk}\frac{\partial x_j}{\partial x_k}}_{\substack{e_{ijk}\delta_{jk}=\\ e_{ijj}=0}}\,p_0\,dV - \int_V e_{ijk}x_j\frac{\partial p_0}{\partial x_k}\,dV = -\int_V e_{ijk}x_j\frac{\partial p_0}{\partial x_k}\,dV \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i\in\{1,2,3\} \end{array}\right. \tag{10.35}$$

and replacing the fundamental equation of hydrostatics (10.25) in (10.35) finally yields

$$\mathbf{M}_E^G = -\int_V (\mathbf{x}\times\nabla p_0)\,dV = -\underbrace{\int_V (\mathbf{x}\times\rho_0\,\mathbf{b})\,dV}_{\mathbf{M}_W^G} = -\mathbf{M}_W^G = \mathbf{0}\,, \tag{10.36}$$

[2] Without loss of generality, the origin of the system of Cartesian axes can be placed at G.

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where $\mathbf{M}_W^G$ is the moment of the weight of the displaced fluid with respect to its center of gravity G, which, considering the definition of the center of gravity, is null. Consequently, the moment of the upward buoyant force E *with respect to the center of gravity of the volume of displaced fluid* is also null. Then, it is concluded that the vertical line of application of the upward buoyant force crosses said center of gravity, as established by the second part of Archimedes' principle.

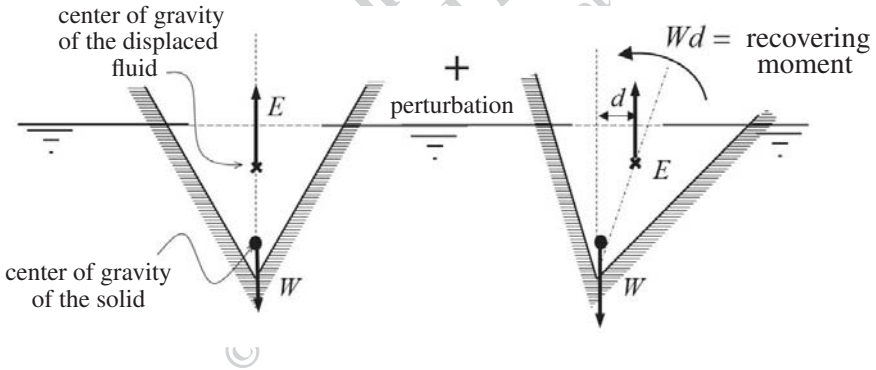
Example 10.1 – *Apply Archimedes' principle to the study of stability of the equilibrium in floating solids to determine how the relative position of the centers of gravity of the solid and the corresponding volume of displaced fluid affect the nature of this equilibrium.*

Solution

Consider a floating medium, in equilibrium, and the following two situations:

a) The center of gravity of the solid (center of thrust) is below the center of gravity of the displaced fluid (center of buoyancy).

In this case, any perturbation (inclination) tends to create a moment $M = Wd$ in the sense that tends to recover the initial state of equilibrium. It is, thus, a case of *stable flotation equilibrium.*



b) The center of gravity of the solid (center of thrust) is above the center of gravity of the displaced fluid (center of buoyancy).

In this case, any perturbation (inclination) tends to create a moment $M = Wd$ in the sense that tends to capsize the floating solid, that is, it tends to move the solid further away from the initial state of equilibrium. It is, thus, a case of *unstable flotation equilibrium*.

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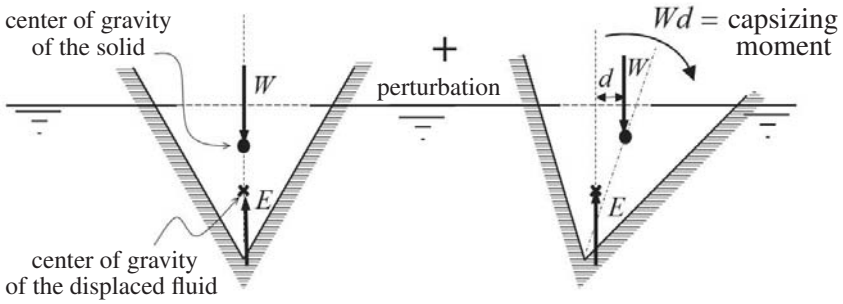

Placing weights (*ballasts*) on the keel of a boat responds to the search of improved flotation stability of this boat.

10.3 Fluid Dynamics: Barotropic Perfect Fluids

In the most common case, the velocity is not uniform nor stationary ($\mathbf{v} \equiv \mathbf{v}(\mathbf{x},t)$), and, therefore, in general, the acceleration will not be null ($\mathbf{a}(\mathbf{x},t) \neq \mathbf{0}$). In consequence, the divergence of the velocity ($\nabla \cdot \mathbf{v} \neq 0$) and the gradient of the velocity $\left(\nabla \otimes \mathbf{v} \overset{not}{=} \nabla \mathbf{v} \neq \mathbf{0}\right)$ will not be null either.

Definition 10.2. A *perfect fluid* is a Newtonian fluid characterized by the fact that the viscosities λ and μ (see (10.14)) are null.

The mechanical constitutive equation (10.14) of a perfect fluid becomes

$$\left.\begin{array}{l} \boldsymbol{\sigma} = -p\,\mathbf{1} + \lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d} \\ \lambda = \mu = 0 \end{array}\right\} \quad \Longrightarrow \quad \boxed{\boldsymbol{\sigma} = -p\,\mathbf{1}} \tag{10.37}$$

$$\Longrightarrow \quad \left\{\begin{array}{l} \nabla \cdot \boldsymbol{\sigma} = -\nabla p \\ \boldsymbol{\sigma} : \mathbf{d} = -p\,\mathbf{1} : \mathbf{d} = -p\,\mathrm{Tr}(\mathbf{d}) \end{array}\right.$$

which results in a hydrostatic stress state[3].

[3] A hydrostatic stress state (the stress tensor is spherical) should not be confused with a hydrostatic motion regime (the velocity is uniform or null).

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Definition 10.3. A *barotropic fluid* is characterized by a kinetic equation of state (10.9) in which the temperature does not intervene.

$$F(\rho, p, \theta) \equiv F(\rho, p) = 0 \quad \Longrightarrow \quad \rho = \rho(p)$$

10.3.1 Equations of the Problems

Taking into account the hypotheses of a perfect and a barotropic fluid, the equations governing a fluid dynamics problem are reduced to:

a) **Mechanical problem**

1) *Mass continuity equation*

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \qquad \text{(1 equation)} \tag{10.38}$$

2) *Balance of linear momentum* (Euler's equation)

$$-\nabla p + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \qquad \text{(3 equations)} \tag{10.39}$$

3) *Kinetic equation of state*

$$\rho = \rho(p) \qquad \text{(1 equation)} \tag{10.40}$$

The mechanical problem is composed of 5 equations and 5 unknowns $\left(\rho(\mathbf{x},t), \mathbf{v}(\mathbf{x},t), p(\mathbf{x},t)\right)$ that can be solved uncoupled from the thermal problem.

b) **Thermal problem**

1) *Fourier's law*

$$\mathbf{q} = -K \nabla \theta \;\Rightarrow\; \nabla \cdot \mathbf{q} = -K \nabla \cdot (\nabla \theta) = -K \nabla^2 \theta \qquad \text{(3 equations)} \tag{10.41}$$

2) *Energy balance*

$$\rho \frac{du}{dt} = \underbrace{-p \nabla \cdot \mathbf{v}}_{\boldsymbol{\sigma} : \mathbf{d}} + \rho r + \underbrace{K \nabla^2 \theta}_{-\nabla \cdot \mathbf{q}} \qquad \text{(1 equation)} \tag{10.42}$$

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3) *Caloric equation of state*

$$u = u(\rho, \theta) \qquad \text{(1 equation)} \qquad (10.43)$$

The thermal problem is defined by 5 equations and 5 unknowns $\big(\mathbf{q}(\mathbf{x},t)$, $\theta(\mathbf{x},t)$, $u(\mathbf{x},t)\big)$ and can be solved *once the mechanical problem has been solved* and the velocity field $\mathbf{v}(\mathbf{x},t)$, the density $\rho(\mathbf{x},t)$ and the pressure $p(\mathbf{x},t)$ are known.

Remark 10.2. A general format of the fluid mechanics problem includes the thermal conductivity K between the viscosities (in a generalized sense) of the problem. The definition of a *perfect fluid* as a *fluid without viscosity* results, in this context, in the cancellation of the thermal conductivity $(K = 0)$, therefore (10.41) leads to $\mathbf{q} = -K\nabla\theta = \mathbf{0}$ and the thermal problem is reduced to the equations (10.42) and (10.43).

10.3.2 Resolution of the Mechanical Problem under Potential Body Forces. Bernoulli's Trinomial

Consider now the mechanical problem for the particular case of *potential body forces* (the body forces derive from a potential ϕ),

$$\boxed{\text{Potential body forces:} \quad \mathbf{b}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t)} \; . \qquad (10.44)$$

In the particular case of a gravitational potential with the line of action along the negative direction of the z-axis, the potential is

$$\phi(x,y,z,t) = g\,z \quad \Longrightarrow \quad \mathbf{b} = -\nabla\phi \overset{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \; . \qquad (10.45)$$

Lemma 10.1. *For a barotropic fluid* $(\rho = \rho(p))$ *there exists a function* $\mathbb{P}(\mathbf{x},t) = \hat{\mathbb{P}}(p(\mathbf{x},t))$ *that satisfies*

$$\nabla p = \rho\,\nabla\mathbb{P} \; .$$

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Proof
Defining the function $\mathbb{P}(\mathbf{x},t)$ as

$$\mathbb{P}(\mathbf{x},t) = \hat{\mathbb{P}}(p(\mathbf{x},t)) = \int_0^p \frac{1}{\rho(\bar{p})} d\bar{p}\,, \tag{10.46}$$

then, it will satisfy

$$\begin{cases} \dfrac{\partial \mathbb{P}(\mathbf{x},t)}{\partial x_i} = \dfrac{\partial \hat{\mathbb{P}}}{\partial p}\dfrac{\partial p}{\partial x_i} \\ \left[\nabla \mathbb{P}\right]_i = \dfrac{\partial \hat{\mathbb{P}}}{\partial p}\left[\nabla p\right]_i = \dfrac{1}{\rho(p)}\left[\nabla p\right]_i \qquad i \in \{1,2,3\} \end{cases} \tag{10.47}$$

leading to

$$\boxed{\nabla \mathbb{P} = \frac{1}{\rho(p)} \nabla p}\,. \tag{10.48}$$

Lemma 10.2. *The convective term of the acceleration can be written as*

$$\mathbf{v}\cdot\nabla\mathbf{v} = 2\boldsymbol{\omega}\times\mathbf{v} + \nabla\left(\frac{1}{2}\mathbf{v}^2\right),$$

where $2\boldsymbol{\omega} = \nabla\times\mathbf{v}$ *is the vorticity vector.*

Proof
Expanding the right-hand term in the Lemma[4],

$$\begin{aligned} \left[\mathbf{v}\cdot\nabla\mathbf{v}\right]_j &= \mathrm{v}_i \frac{\partial \mathrm{v}_j}{\partial x_i} = \mathrm{v}_i \underbrace{\left(\frac{\partial \mathrm{v}_j}{\partial x_i} - \frac{\partial \mathrm{v}_i}{\partial x_j}\right)}_{2w_{ji}} + \mathrm{v}_i \frac{\partial \mathrm{v}_i}{\partial x_j} = 2\,\mathrm{v}_i \underbrace{w_{ji}}_{-w_{ij}} + \mathrm{v}_i \frac{\partial \mathrm{v}_i}{\partial x_j} = \\ &= -2\,\mathrm{v}_i w_{ij} + \mathrm{v}_i \frac{\partial \mathrm{v}_i}{\partial x_j} = 2\underbrace{e_{ijk}\,\mathrm{v}_i\,\omega_k}_{e_{jki}\,\mathrm{v}_i\,\omega_k} + \mathrm{v}_i \frac{\partial \mathrm{v}_i}{\partial x_j} = \\ &= \underbrace{2\,e_{jki}\,\mathrm{v}_i\,\omega_k}_{\left[2\boldsymbol{\omega}\times\mathbf{v}\right]_j} + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\underbrace{\mathrm{v}_i\mathrm{v}_i}_{\mathbf{v}\cdot\mathbf{v}=\mathrm{v}^2}\right) = \left[2\boldsymbol{\omega}\times\mathbf{v}\right]_j + \left[\nabla\left(\frac{1}{2}\mathrm{v}^2\right)\right]_j\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \in \{1,2,3\} \end{aligned} \tag{10.49}$$

[4] The following results, previously obtained in Chapter 2, are used here:
$w_{ji} = -w_{ij} = \left[\nabla^a \mathbf{v}\right]_{ji} = \left(\partial \mathrm{v}_j/\partial x_i - \partial \mathrm{v}_i/\partial x_j\right)/2$, $w_{ij} = -e_{ijk}\omega_k$ and $\mathrm{v}^2 = |\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$.

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which leads to

$$\boxed{\mathbf{v}\cdot\nabla\mathbf{v} = 2\boldsymbol{\omega}\times\mathbf{v} + \nabla\left(\frac{1}{2}\mathrm{v}^2\right)}\,. \tag{10.50}$$

Rearranging now Euler's equation (10.39),

$$-\nabla p + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} \qquad \Longrightarrow \qquad -\frac{1}{\rho}\nabla p + \mathbf{b} = \frac{d\mathbf{v}}{dt}\,, \tag{10.51}$$

and replacing (10.45) and (10.48) in (10.51) produces

$$-\nabla\mathbb{P} - \nabla\phi = \frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{v} + \nabla\left(\frac{1}{2}\mathrm{v}^2\right)\,, \tag{10.52}$$

where the result (10.50) has been taken into account. Finally, (10.52) is rewritten as

$$-\left(\nabla\mathbb{P} + \nabla\phi + \nabla\left(\frac{1}{2}\mathrm{v}^2\right)\right) = \frac{\partial\mathbf{v}}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{v}\,. \tag{10.53}$$

Equation of motion of a barotropic perfect fluid under potential body forces

$$-\nabla\underbrace{\left(\mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2\right)}_{\text{Bernoulli's trinomial}} = \frac{\partial\mathbf{v}}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{v} \tag{10.54}$$

Equation (10.54) is the particular form adopted by the balance of linear momentum (Euler's equation (10.39)) in barotropic perfect fluids subjected to potential body forces.

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10.3.3 Solution in a Steady-State Regime

The solution to the mechanical problem defined by (10.38) to (10.40) has, in general, a *transient regime*, in which the spatial description of the mechanical variables evolves along time, and a *steady-state regime*, in which said spatial description is, approximately, constant along time (see Figure 10.8).

Consider now the equation of motion (10.54) in a *steady-state regime*,

$$\frac{\partial\mathbf{v}}{\partial t} = \mathbf{0} \qquad \Longrightarrow \qquad -\nabla\left(\mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2\right) = 2\boldsymbol{\omega}\times\mathbf{v}\,, \tag{10.55}$$

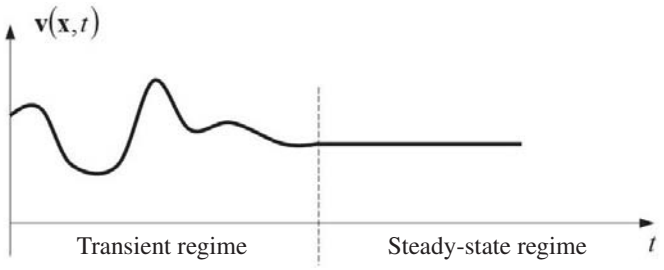

Figure 10.8: Transient and steady-state regimes.

and a streamline[5] $\Gamma:\ \mathbf{x}=\mathbf{x}(s)$ parametrized in terms of its arc-length s (see Figure 10.9). Projecting (multiplying) equation (10.53) in the direction tangent to the streamline, $\mathbf{t}$, results in

$$-\nabla\underbrace{\left(\mathbb{P}+\phi+\frac{1}{2}\mathrm{v}^2\right)}_{M(\mathbf{x})}=2\boldsymbol{\omega}\times\mathbf{v}\implies-(\nabla M)\cdot\underbrace{\mathbf{v}}_{\frac{d\mathbf{x}}{ds}\frac{ds}{dt}}=\underbrace{(2\boldsymbol{\omega}\times\mathbf{v})\cdot\mathbf{v}}_{=\mathbf{0}}=\mathbf{0} \tag{10.56}$$

$$\left.\begin{aligned}\nabla M(\mathbf{x}(s))\cdot\frac{d\mathbf{x}}{ds}&=\frac{dM}{ds}=0\\ \frac{\partial M(\mathbf{x}(s))}{\partial x_i}\frac{dx_i}{ds}&=\frac{dM}{ds}=0\end{aligned}\right\}\ \forall\mathbf{x}\in\Gamma\implies\boxed{M(\mathbf{x})=const.\quad\forall\mathbf{x}\in\Gamma} \tag{10.57}$$

and (10.57) is written as

$$\boxed{\left[\mathbb{P}+\phi+\frac{1}{2}\mathrm{v}^2\right](\mathbf{x})=const.\quad\forall\mathbf{x}\in\Gamma}\ , \tag{10.58}$$

which establishes that *Bernoulli's trinomial remains constant along a same streamline* Γ.

Remark 10.3. Note that (10.58) is no longer a partial differential equation but a (scalar) algebraic equation, already integrated. This equation allows, thus, determining one of the unknowns of the mechanical problem once the others are known.

[5] In a steady-state (stationary) regime, trajectories and streamlines coincide.

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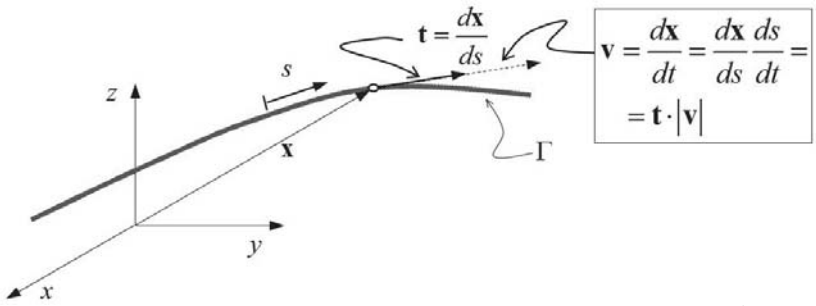

Figure 10.9: Parametrized streamline.

10.3.3.1 Solution in Steady-State Regime for an Incompressible Fluid under Gravitational Forces

Consider now the particular case of a barotropic fluid with the following characteristics:

a) **The fluid is *incompressible***

$$\rho = \rho(p) = \rho_0 = const. \tag{10.59}$$

In this case, the function $\mathbb{P}(p)$ in (10.46) can be integrated as follows.

$$\mathbb{P}(\mathbf{x},t) = \int_0^p \frac{1}{\rho(\bar{p})} d\bar{p} = \frac{1}{\rho_0} \int_0^p d\bar{p} = \frac{p}{\rho_0} \tag{10.60}$$

b) **The body forces are *gravitational***

In accordance with (10.45),

$$\phi = gz \qquad \mathbf{b} = -\nabla\phi \overset{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \tag{10.61}$$

Introducing (10.60) and (10.61) in Bernouilli's trinomial (10.58) yields

$$\frac{p}{\rho_0} + gz + \frac{1}{2}\mathrm{v}^2 = const. \implies z + \frac{p}{\rho_0 g} + \frac{1}{2}\frac{\mathrm{v}^2}{g} \overset{def}{=} H = const. \qquad \forall \mathbf{x} \in \Gamma \tag{10.62}$$

The terms in (10.61) have dimensions of length (height) and may be interpreted in the following manner.

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Bernoulli's theorem

$$\underbrace{z}_{\text{elevation}} + \underbrace{\frac{p}{\rho_0 g}}_{\substack{\text{pressure}\\ \text{head}}} + \underbrace{\frac{1}{2}\frac{v^2}{g}}_{\substack{\text{velocity}\\ \text{head}}} \stackrel{def}{=} \underbrace{H}_{\substack{\text{total}\\ \text{height}}} = const. \qquad \forall \mathbf{x} \in \Gamma \tag{10.63}$$

Remark 10.4. The expression in (10.63) constitutes the so-called **Bernoulli's theorem** (for an incompressible perfect fluid under gravitational forces and in steady-state regime), which establishes that *the sum of the elevation, the pressure head and the velocity head* is constant *in every point belonging to a same streamline* (see Figure 10.10).

Remark 10.5. In engineering, water is generally considered an incompressible and perfect fluid, and the science that studies it is named *hydraulics*. Since, in general, the body forces are of the gravitational type, Bernoulli's Theorem is generally applicable in the *resolution of steady-state problems in hydraulics*.

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Theory and Problems
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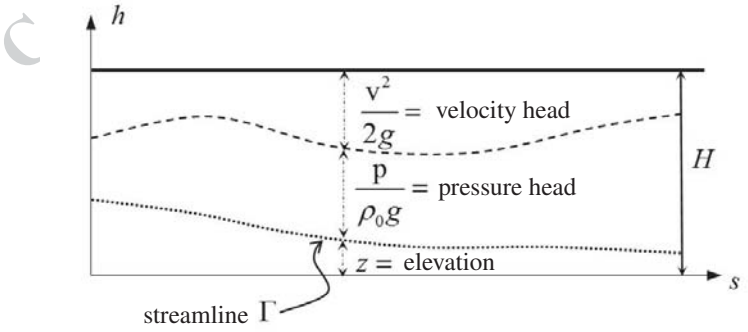

Figure 10.10: Physical interpretation of Benoulli's theorem.

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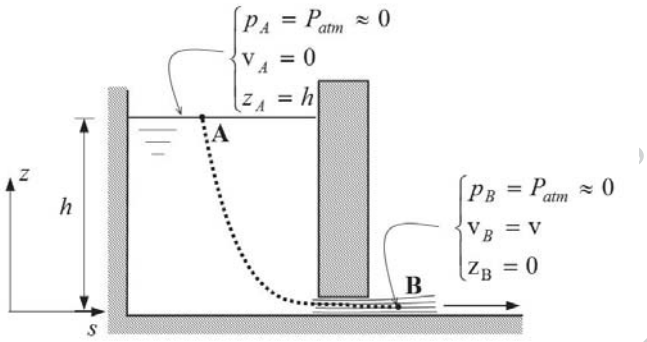
Example 10.2 – *Determine the velocity of the water exiting the tank through a small lateral hole placed at a distance h below the top surface of the water. Consider the top of the tank is open and neglect the atmospheric pressure. Assume a steady-state regime.*



Solution

The fluid in this problem (water) is an incompressible perfect fluid in steady-state regime under gravitational forces and, thus, Bernoulli's theorem can be applied.
Consider a streamline originating at point **A** of the water surface and ending at point **B** of the exit hole (shown in the figure above). Applying Bernoulli's theorem between points **A** and **B**, and taking into account that the velocity of the free surface in the tank is practically null and that its cross-section is much larger than that of the exit hole, then

$$\underbrace{z_A}_{=h} + \underbrace{\frac{p_A}{\rho_0 g}}_{=0} + \underbrace{\frac{1}{2}\frac{v_A^2}{g}}_{=0} = \underbrace{z_B}_{=0} + \underbrace{\frac{p_B}{\rho_0 g}}_{=0} + \frac{1}{2}\frac{v_B^2}{g}$$

$$h + 0 + 0 = 0 + 0 + \frac{1}{2}\frac{v^2}{g} \quad \Longrightarrow \quad \boxed{v = \sqrt{2gh}} \, .$$

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10.3.4 Solution in Transient Regime

In a transient regime, the mechanical variables (in their spatial description) are time-dependent (see Figure 10.8). The starting point to solve the problem is the balance of linear momentum (10.54),

$$\boxed{-\nabla\left(\mathbb{P}+\phi+\frac{1}{2}\mathrm{v}^2\right)=\frac{\partial \mathbf{v}}{\partial t}+2\boldsymbol{\omega}\times\mathbf{v}\,.} \tag{10.64}$$

In some cases, the solution to this equation in transient regime is particularly simple. In the following subsections, several of these cases will be studied.

10.3.4.1 (Irrotational) Potential Flow

Consider the case of
- a perfect fluid
- with potential body forces
- and irrotational flow.

Definition 10.4. The motion (or flow) of a fluid is said to be *irrotational* (or potential) if the rotational of the velocity field is null at any point of this fluid.

In other words, an irrotational flow has a null vorticity vector.

$$\text{Irrotational flow} \quad \begin{cases} \nabla\times\mathbf{v}(\mathbf{x},t)=\mathbf{0} \\ \boldsymbol{\omega}(\mathbf{x},t)=\dfrac{1}{2}\nabla\times\mathbf{v}(\mathbf{x},t)=\mathbf{0} \end{cases} \quad \forall\mathbf{x}\ \ \forall t \tag{10.65}$$

If the flow is irrotational, it is inferred from (10.65) that there exists a scalar function (denoted as velocity potential $\chi(\mathbf{x},t)$) that satisfies[6]

$$\mathbf{v}(\mathbf{x},t)=\nabla\chi(\mathbf{x},t)\ . \tag{10.66}$$

Note that, in this case, the *vector* field $\mathbf{v}(\mathbf{x},t)$ is determined in terms of the *scalar* velocity potential $\chi(\mathbf{x},t)$ (which becomes the main unknown of the problem). Replacing the conditions (10.65) and (10.66) in (10.64) yields

$$-\nabla\left(\mathbb{P}+\phi+\frac{1}{2}\mathrm{v}^2\right)=\frac{\partial \mathbf{v}}{\partial t}+2\underbrace{\boldsymbol{\omega}}_{=\mathbf{0}}\times\mathbf{v}=\frac{\partial \mathbf{v}}{\partial t}=\frac{\partial}{\partial t}\Big(\nabla\chi(\mathbf{x},t)\Big)=\nabla\left(\frac{\partial\chi}{\partial t}\right)\ \Longrightarrow \tag{10.67}$$

[6] It can be proven that, given an irrotational vector field $\mathbf{v}(\mathbf{x},t)$, that is, a vector field that satisfies $\nabla\times\mathbf{v}=\mathbf{0}$, there exists a scalar function $\chi(\mathbf{x},t)$ (potential function) such that $\mathbf{v}=\nabla\chi(\mathbf{x},t)$. Obviously, since $\nabla\times\nabla(\bullet)\equiv\mathbf{0}$, then $\nabla\times\mathbf{v}=\nabla\times\nabla\chi(\mathbf{x},t)=\mathbf{0}$ is satisfied.

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$$\boxed{\begin{array}{ll} \nabla \underbrace{\left(\mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2 + \frac{\partial \chi}{\partial t} \right)}_{M(\mathbf{x},t)} = \nabla M(\mathbf{x},t) = \mathbf{0} & \\ & \forall \mathbf{x} \ \forall t \\ \dfrac{\partial M(\mathbf{x},t)}{\partial x_i} = 0 \qquad i \in \{1,2,3\} & \end{array}} \tag{10.68}$$

This equation can be trivially integrated, resulting in

$$M(\mathbf{x},t) = \mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2 + \frac{\partial \chi}{\partial t} = \varphi(t) \ . \tag{10.69}$$

Defining a modified velocity potential $\bar{\chi}(\mathbf{x},t)$ of the form

$$\bar{\chi}(\mathbf{x},t) \stackrel{def}{=} \chi(\mathbf{x},t) - \int_0^t \varphi(\tau)\, d\tau \quad \Rightarrow \quad \begin{cases} \nabla \bar{\chi} = \nabla \chi = \mathbf{v}(\mathbf{x},t) \\ \dfrac{\partial \bar{\chi}}{\partial t} = \dfrac{\partial \chi}{\partial t} - \varphi(t) \end{cases} \tag{10.70}$$

and replacing (10.70) in (10.69) produces

$$\mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2 + \underbrace{\frac{\partial \chi}{\partial t} - \varphi(t)}_{\frac{\partial \bar{\chi}}{\partial t}} = 0 \implies \boxed{\mathbb{P} + \phi + \frac{1}{2}(\nabla \bar{\chi})^2 + \frac{\partial \bar{\chi}}{\partial t} = 0 \quad \forall \mathbf{x} \ \forall t} \ , \tag{10.71}$$

which is the differential equation of *hydraulic transients*.

The mechanical problem is then defined by:

1) *Mass continuity equation*

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = \frac{d\rho}{dt} + \rho \underbrace{\nabla \cdot (\nabla \bar{\chi})}_{\nabla^2 \bar{\chi}} = 0 \quad \Longrightarrow \quad \boxed{\frac{d\rho}{dt} + \rho \nabla^2 \bar{\chi} = 0} \tag{10.72}$$

2) *Balance of linear momentum (hydraulic transients equation)*

$$\boxed{\mathbb{P}(\rho, p) + \phi + \frac{1}{2}(\nabla \bar{\chi})^2 + \frac{\partial \bar{\chi}}{\partial t} = 0 \qquad \forall \mathbf{x} \ \forall t} \tag{10.73}$$

3) *Kinetic equation of state*

$$\rho = \rho(p) \tag{10.74}$$

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These constitute a system of 3 scalar equations and 3 unknowns $(p(\mathbf{x},t)$, $\rho(\mathbf{x},t)$ and $\bar{\chi}(\mathbf{x},t))$ that can be integrated in the $\mathbb{R}^3 \times \mathbb{R}^+$ domain. Once the potential $\bar{\chi}(\mathbf{x},t)$ is known, the velocity field is obtained through

$$\mathbf{v}(\mathbf{x},t) = \nabla \bar{\chi}(\mathbf{x},t) \ . \tag{10.75}$$

10.3.4.2 Incompressible and Potential Flow

Consider the case of

- a perfect fluid
- with potential body forces,
- irrotational flow
- and incompressible flow.

Since the flow is incompressible, (10.46) and (10.72) allow determining[7]

$$\frac{d\rho}{dt} = 0 \quad \Longrightarrow \quad \rho = \rho_0 \quad \Longrightarrow \quad \begin{cases} \mathbb{P}(p) = \displaystyle\int_0^p \frac{1}{\rho(\bar{p})} d\bar{p} = \frac{p}{\rho_0} \\ \nabla^2 \bar{\chi} \overset{not}{=} \Delta \bar{\chi} = 0 \end{cases} \tag{10.76}$$

and the mechanical problem (10.72) to (10.74) is reduced to:

1) *Mass continuity equation*

$$\boxed{\Delta \bar{\chi} = \frac{\partial^2 \bar{\chi}}{\partial x_i \partial x_i} = 0} \tag{10.77}$$

2) *Balance of linear momentum (hydraulic transients equation)*

$$\boxed{\frac{p}{\rho_0} + \phi + \frac{1}{2}(\nabla \bar{\chi})^2 + \frac{\partial \bar{\chi}}{\partial t} = 0 \qquad \forall \mathbf{x} \ \forall t} \tag{10.78}$$

These constitute a system of *2 scalar equations* and 2 unknowns $(p(\mathbf{x},t)$ and $\bar{\chi}(\mathbf{x},t))$ that can be integrated in the $\mathbb{R}^3 \times \mathbb{R}^+$ domain. In a steady-state regime, the term $\partial \bar{\chi}/\partial t = 0$ and any time derivative in the system disappears, such that the problem can be integrated in $\mathbb{R}^3$.

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[7] Here, the differential operator named Laplace operator or *Laplacian of* $(\bullet)$ is defined as $\Delta(\bullet) = \nabla \cdot \nabla(\bullet) \overset{not}{=} \nabla^2(\bullet) = \partial^2(\bullet)/\partial x_i \partial x_i$.

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10.4 Fluid Dynamics: (Newtonian) Viscous Fluids

Consider now the general problem described by (10.1) to (10.9),

$$
\begin{aligned}
&\frac{d\rho}{dt}+\rho\nabla\cdot\mathbf{v}=0 && \text{Mass continuity equation} && (1\text{ eqn.})\\
&\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\frac{d\mathbf{v}}{dt} && \text{Balance of linear momentum} && (3\text{ eqns.})\\
&\rho\frac{du}{dt}=\boldsymbol{\sigma}:\mathbf{d}+\rho\, r-\nabla\cdot\mathbf{q} && \text{Energy balance} && (1\text{ eqn.})\\
&\boldsymbol{\sigma}=-p\mathbf{1}+\lambda\,\mathrm{Tr}\,(\mathbf{d})\,\mathbf{1}+2\mu\,\mathbf{d} && \text{Mechanical constitutive equation} && (6\text{ eqns.})\\
&s=s\,(\mathbf{d},\theta,\rho) && \text{Entropy constitutive equation} && (1\text{ eqn.})\\
&\mathbf{q}=-K\nabla\theta && \text{Heat conduction equation} && (3\text{ eqns.})\\
&u=u\,(\rho,\theta) && \text{Caloric equation of state} && (1\text{ eqn.})\\
&F\,(\rho,p,\theta)=0 && \text{Kinetic equation of state} && (1\text{ eqn.})
\end{aligned}
\tag{10.79}
$$

which constitute a system of 17 equations and 17 unknowns. This system is too large to be treated efficiently and a reduced system of equations that allows a simpler resolution will be sought.

10.4.1 Navier-Stokes Equation

The Navier-Stokes equation is essentially the balance of linear momentum of (10.79) expressed solely in terms of the velocity field $\mathbf{v}\,(\mathbf{x},t)$ and the pressure $p\,(\mathbf{x},t)$.

Lemma 10.3. *The divergence of the strain rate tensor* $\mathbf{d}\,(\mathbf{x},t)$ *is related to the velocity field* $\mathbf{v}\,(\mathbf{x},t)$ *by*

$$\nabla\cdot\mathbf{d}=\frac{1}{2}\Delta\mathbf{v}+\frac{1}{2}\nabla\,(\nabla\cdot\mathbf{v})\ .$$

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Proof

$$\left[\nabla\cdot\mathbf{d}\right]_j = \frac{\partial}{\partial x_i} d_{ij} = \frac{\partial}{\partial x_i}\left(\frac{1}{2}\left(\frac{\partial \mathrm{v}_i}{\partial x_j}+\frac{\partial \mathrm{v}_j}{\partial x_i}\right)\right) = \frac{1}{2}\frac{\partial^2 \mathrm{v}_i}{\partial x_i \partial x_j}+\frac{1}{2}\frac{\partial^2 \mathrm{v}_j}{\partial x_i \partial x_i} =$$

$$= \frac{1}{2}\frac{\partial}{\partial x_j}\underbrace{\frac{\partial \mathrm{v}_i}{\partial x_i}}_{\nabla\cdot\mathbf{v}} + \frac{1}{2}\underbrace{\frac{\partial^2 \mathrm{v}_j}{\partial x_i \partial x_i}}_{\Delta \mathrm{v}_j} = \frac{1}{2}\underbrace{\frac{\partial}{\partial x_j}(\nabla\cdot\mathbf{v})}_{\left[\nabla(\nabla\cdot\mathbf{v})\right]_j} + \frac{1}{2}\underbrace{\Delta \mathrm{v}_j}_{\left[\Delta\mathbf{v}\right]_j} =$$

$$= \left[\frac{1}{2}\Delta\mathbf{v}+\frac{1}{2}\nabla(\nabla\cdot\mathbf{v})\right]_j \qquad j\in\{1,2,3\} \tag{10.80}$$

$$\boxed{\nabla\cdot\mathbf{d} = \frac{1}{2}\Delta\mathbf{v}+\frac{1}{2}\nabla(\nabla\cdot\mathbf{v})} \tag{10.81}$$

Lemma 10.4. *Given a scalar function $\alpha(\mathbf{x},t)$, the following is satisfied.*

$$\nabla\cdot(\alpha\mathbf{1}) = \nabla\alpha$$

Proof

$$\left[\nabla\cdot(\alpha\mathbf{1})\right]_i = \frac{\partial(\alpha\delta_{ij})}{\partial x_j} = \delta_{ij}\frac{\partial\alpha}{\partial x_j} = \frac{\partial\alpha}{\partial x_i} = \left[\nabla\alpha\right]_i \qquad i\in\{1,2,3\} \tag{10.82}$$

$$\boxed{\nabla\cdot(\alpha\mathbf{1}) = \nabla\alpha} \tag{10.83}$$

Replacing the mechanical constitutive equation of (10.79) into the balance of linear momentum of (10.79), and taking into account (10.81) and (10.83) leads to

$$\left.\begin{aligned}&\boldsymbol{\sigma} = -p\mathbf{1}+\lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1}+2\mu\,\mathbf{d}\\&\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt}\end{aligned}\right\} \quad\Longrightarrow$$

$$\begin{cases}\nabla\cdot\boldsymbol{\sigma} = -\nabla p+\lambda\underbrace{\nabla(\mathrm{Tr}(\mathbf{d}))}_{\nabla(\nabla\cdot\mathbf{v})}+\mu\Delta\mathbf{v}+\mu\nabla(\nabla\cdot\mathbf{v})\\ \nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b} = -\nabla p+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{v})+\mu\Delta\mathbf{v}+\rho\mathbf{b} = \rho\dfrac{d\mathbf{v}}{dt}\end{cases} \tag{10.84}$$

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which results in the Navier-Stokes equation.

Navier-Stokes equation

$$-\nabla p+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{v})+\mu\Delta\mathbf{v}+\rho\mathbf{b}=\rho\frac{d\mathbf{v}}{dt}$$
$$-\frac{\partial p}{\partial x_i}+(\lambda+\mu)\frac{\partial^2 \mathrm{v}_j}{\partial x_i\partial x_j}+\mu\frac{\partial^2 \mathrm{v}_i}{\partial x_j\partial x_j}+\rho b_i=\rho\frac{d\mathrm{v}_i}{dt};\ \ i\in\{1,2,3\} \tag{10.85}$$

10.4.2 Energy Equation

The aim is to eliminate $\boldsymbol{\sigma}$ and $\mathbf{q}$ from the energy balance of (10.79) by replacing in this equation the mechanical constitutive equation and the entropy equation of (10.79). To this aim, the definition of stress power in a Newtonian fluid (see Chapter 9) is recovered,

$$\boldsymbol{\sigma}:\mathbf{d}=W_R+2W_D=-p\nabla\cdot\mathbf{v}+\mathcal{K}\,\mathrm{Tr}^2(\mathbf{d})+2\mu\,\mathbf{d}':\mathbf{d}', \tag{10.86}$$

where $\mathbf{d}'$ is the deviatoric part of the strain rate tensor. Fourier's law is also recovered,

$$\mathbf{q}=-K\nabla\theta \quad\Longrightarrow\quad \nabla\cdot\mathbf{q}=-\nabla\cdot(K\nabla\theta)\ . \tag{10.87}$$

Replacing now in the energy balance of (10.79) yields

$$\rho\frac{du}{dt}=\boldsymbol{\sigma}:\mathbf{d}+\rho r-\nabla\cdot\mathbf{q} \quad\Longrightarrow \tag{10.88}$$

Energy equation

$$\rho\frac{du}{dt}=-p\nabla\cdot\mathbf{v}+\rho r+\nabla\cdot(K\nabla\theta)+\underbrace{\mathcal{K}\,\mathrm{Tr}^2(\mathbf{d})+2\mu\,\mathbf{d}':\mathbf{d}'}_{2W_D}$$
$$\rho\frac{du}{dt}=-p\frac{\partial \mathrm{v}_i}{\partial x_i}+\rho r+\frac{\partial}{\partial x_i}\left(K\frac{\partial\theta}{\partial x_i}\right)+\mathcal{K}\left(\frac{\partial \mathrm{v}_i}{\partial x_i}\right)^2+2\mu\,d'_{ij}d'_{ij} \tag{10.89}$$

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10.4.3 Governing Equations of the Fluid Mechanics Problem

Considering the simplified versions of the balance of linear momentum (Navier-Stokes equation (10.85)) and the energy balance (energy equation (10.89)) the problem defined in (10.79) can be reduced to the following system of 7 PDEs and 7 unknowns ($\rho(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)$, $p(\mathbf{x},t)$, $u(\mathbf{x},t)$, $\theta(\mathbf{x},t)$), which must be solved in the $\mathbb{R}^3 \times \mathbb{R}^+$ domain.

$$
\begin{aligned}
&\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 && \text{Mass continuity equation} && (1\ \text{eqn.})\\
&-\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \\
&\qquad +\mu\Delta\mathbf{v} + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} && \text{Balance of linear momentum (Navier-Stokes)} && (3\ \text{eqns.})\\
&\rho\frac{du}{dt} = -p\nabla\cdot\mathbf{v} + \rho r + \nabla\cdot(K\nabla\theta) + \\
&\qquad +\mathcal{K}\,\mathrm{Tr}^2(\mathbf{d}) + 2\mu\,\mathbf{d}' : \mathbf{d}' && \text{Energy balance} && (1\ \text{eqn.})\\
&u = u(\rho,\theta) && \text{Caloric equation of state} && (1\ \text{eqn.})\\
&F(\rho,p,\theta) = 0 && \text{Kinetic equation of state} && (1\ \text{eqn.})
\end{aligned}
\tag{10.90}
$$

In the particular case of a *barotropic regime* ($\rho = \rho(p)$), the mechanical part can be uncoupled from the thermal part in the set of equations of (10.79), resulting in the *mechanical problem* defined by the following system of 5 equations and 5 unknowns ($\rho(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)$, $p(\mathbf{x},t)$).

$$
\begin{aligned}
&\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 && \text{Mass continuity equation} && (1\ \text{eqn.})\\
&-\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \\
&\qquad +\mu\Delta\mathbf{v} + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} && \text{Balance of linear momentum (Navier-Stokes)} && (3\ \text{eqns.})\\
&\rho = \rho(p) && \text{Kinetic equation of state (barotropic)} && (1\ \text{eqn.})
\end{aligned}
\tag{10.91}
$$

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10.4.4 Physical Interpretation of the Navier-Stokes and Energy Equations

Each of the terms in the Navier-Stokes equation (10.85),

$$\begin{cases} -\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\Delta\mathbf{v} + \rho\mathbf{b} - \rho\underbrace{\dfrac{d\mathbf{v}}{dt}}_{\mathbf{a}} = \mathbf{0} \\ -\dfrac{\partial p}{\partial x_i} + \Big[(\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\Delta\mathbf{v}\Big]_i + \rho\, b_i - \rho\, a_i = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i\in\{1,2,3\} \end{cases} \tag{10.92}$$

can be interpreted as a component of the system of forces (per unit of volume) that acts on a volume differential of the fluid in motion as follows.

$$\underbrace{-\nabla p}_{\substack{\text{forces due to}\\ \text{the pressure}\\ \text{gradient}}} - \underbrace{-\Big((\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\Delta\mathbf{v}\Big)}_{\substack{\text{viscous forces exerted by}\\ \text{the contact between particles}\\ (=0 \text{ when } \lambda=\mu=0)}} + \underbrace{\rho\mathbf{b}}_{\substack{\text{body}\\ \text{forces}}} + \underbrace{\rho\mathbf{a}}_{\substack{\text{inertial}\\ \text{forces}}} = \mathbf{0} \tag{10.93}$$

Figure 10.11 shows the projection of each of these components in the x_i-direction.

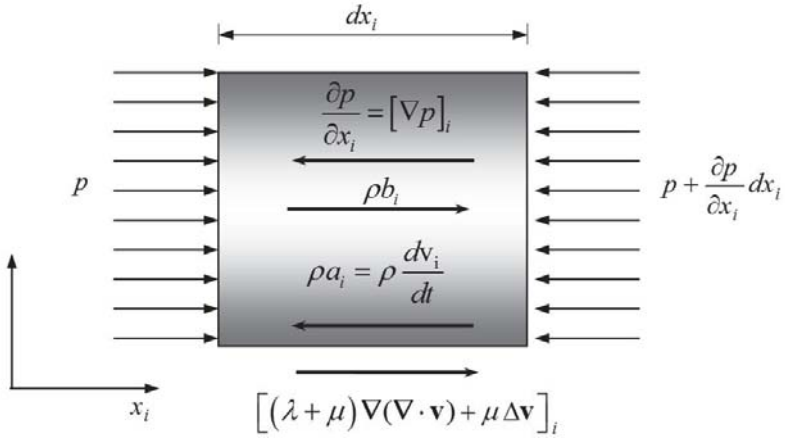

Figure 10.11: Projection of the components of the Navier-Stokes equation in the x_i-direction.

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Each of the terms in the energy equation (10.89) can also be given a physical interpretation, as indicated in Table 10.1.

$$\rho \frac{du}{dt} = -p\nabla\cdot\mathbf{v} + \rho\, r + \nabla\cdot(k\nabla\theta) + \underbrace{\mathcal{K}\,\mathrm{Tr}^2(\mathbf{d}) + 2\mu\,\mathbf{d}':\mathbf{d}'}_{2W_D}$$

$\rho \dfrac{du}{dt}$	$= \dfrac{\text{variation of internal energy}}{\text{unit of volume and unit of time}}$
$\nabla\cdot\mathbf{v} = \dfrac{\frac{d(dV)}{dt}}{dV}$	$= \dfrac{\text{variation of volume}}{\text{unit of volume and unit of time}}$
$-p\nabla\cdot\mathbf{v} = -\dfrac{p\frac{d(dV)}{dt}}{dV}$	$= \dfrac{\text{mechanical work of the thermodynamic pressure}}{\text{unit of volume and unit of time}}$ (see Figure 10.12 and footnote[8])
$\rho\, r + \nabla\cdot(k\nabla\theta)$	$= \dfrac{\text{heat generated by the internal sources and conduction}}{\text{unit of volume and unit of time}}$
$2W_D = \boldsymbol{\sigma}_D : \mathbf{d}$	$= \dfrac{\text{dissipative power}}{} = \dfrac{\text{mechanical work of the viscous forces}}{\text{unit of volume and unit of time}}$

Table 10.1: Physical interpretation of the energy equation.

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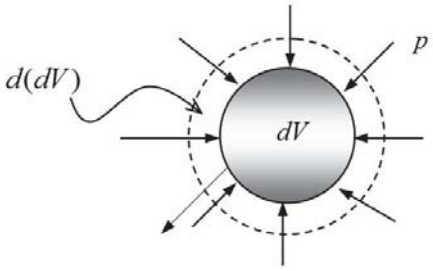

Figure 10.12: Mechanical work of the thermodynamic pressure.

[8] Here, the relation $d(dV)/dt = (\nabla\cdot\mathbf{v})\,dV$ is used (see Section 2.14.3 in Chapter 2).

10.4.5 Reduction of the General Problem to Particular Cases

The governing equations in fluid mechanics (10.90) can be simplified for certain cases which are of particular interest in engineering applications.

10.4.5.1 Incompressible Fluids

In this case,

$$\left.\begin{aligned} &\frac{d\rho}{dt} = 0 \\ &\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \end{aligned}\right\} \quad \Longrightarrow \quad \left\{\begin{aligned} &\rho = \rho_0 = const. \\ &\nabla \cdot \mathbf{v} = \mathrm{Tr}\,(\mathbf{d}) = 0 \end{aligned}\right. \tag{10.94}$$

and introducing (10.94) in (10.90) results in the governing equations detailed in Table 10.2.

Mechanical Problem	Mass continuity equation	$\nabla \cdot \mathbf{v} = 0$
	Navier-Stokes equation	$-\nabla p + \mu \Delta \mathbf{v} + \rho_0 \mathbf{b} = \rho_0 \dfrac{d\mathbf{v}}{dt}$
Thermal Problem	Energy balance	$\rho_0 \dfrac{du}{dt} = \rho_0 r + \nabla \cdot (K \nabla \theta) + 2\mu\, \mathbf{d}' : \mathbf{d}'$
	Caloric equation of state	$u = u(\rho_0, \theta)$
Constitutive equation		$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\, \mathbf{d}$

Table 10.2: Governing equations in incompressible Newtonian fluids

10.4.5.2 Fluids with Null Bulk Viscosity (Stokes Fluids)

In this case,

$$\mathcal{K} = \lambda + \frac{2}{3}\mu = 0 \quad \Longrightarrow \quad \lambda = -\frac{2}{3}\mu \quad \Longrightarrow \quad \lambda + \mu = \frac{1}{3}\mu \tag{10.95}$$

$$2W_D = \underbrace{\mathcal{K}}_{=0} \mathrm{Tr}^2\,(\mathbf{d}) + 2\mu\, \mathbf{d}' : \mathbf{d}' = 2\mu\, \mathbf{d}' : \mathbf{d}' \tag{10.96}$$

and replacing (10.95) and (10.96) in (10.90) yields the governing equations given in Table 10.3.

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Mass continuity equation	$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0$
Navier-Stokes equation	$-\nabla p + \frac{1}{3}\mu \nabla (\nabla \cdot \mathbf{v}) + \mu \Delta \mathbf{v} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt}$
Energy balance	$\rho \frac{du}{dt} = -p \nabla \cdot \mathbf{v} + \rho r + \nabla \cdot (K \nabla \theta) + 2\mu \, \mathbf{d}' : \mathbf{d}'$
Caloric equation of state	$u = u(\rho, \theta)$
Kinetic equation of state	$F(\rho, p, \theta) = 0$
Constitutive equation	$\boldsymbol{\sigma} = -p\mathbf{1} - \frac{2}{3}\mu \, \mathrm{Tr}(\mathbf{d})\mathbf{1} + 2\mu \, \mathbf{d}$

Table 10.3: Governing equations in Stokes fluids.

10.4.5.3 Perfect Fluids

Perfect fluids have null viscosity, $\lambda = \mu = \mathcal{K} = 0$, and no heat conductivity, $K = 0$. Introducing these conditions in (10.90) results in the problem shown in Table 10.4.

Mass continuity equation	$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0$
Euler's equation	$-\nabla p + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt}$
Energy balance	$\rho \frac{du}{dt} = -p \nabla \cdot \mathbf{v} + \rho r$
Caloric equation of state	$u = u(\rho, \theta)$
Kinetic equation of state	$F(\rho, p, \theta) = 0$
Constitutive equation	$\boldsymbol{\sigma} = -p\mathbf{1}$

Table 10.4: Governing equations in perfect fluids.

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10.4.5.4 Hydrostatics

In this case, the following conditions apply (see (10.20)):

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{0}\,, \quad \nabla\cdot\mathbf{v} = 0\,, \quad \rho = \rho_0\,, \quad p = p_0 \quad \text{and} \quad \boldsymbol{\sigma} = -p_0\mathbf{1}\,, \tag{10.97}$$

and, thus, (10.90) is reduced to the equations described in Table 10.5.

Mechanical Problem	Hydrostatics fundamental equation	$-\nabla p_0 + \rho_0\,\mathbf{b} = \mathbf{0}$
Thermal Problem	Energy balance	$\rho_0 \dfrac{du}{dt} = \rho_0\, r + \nabla\cdot(K\nabla\theta)$
	Caloric equation of state	$u = u(\rho_0, \theta)$
Constitutive equation		$\boldsymbol{\sigma} = -p_0\,\mathbf{1}$

Table 10.5: Governing equations in hydrostatics.

10.5 Boundary Conditions in Fluid Mechanics

The governing equations of the fluid mechanics problem presented in the previous sections require adequate boundary conditions to be solved correctly. In general, the *spatial (or Eulerian) description* is used in fluid mechanics problems, and a specific *control volume* (fixed in space) is analyzed, on whose boundary the aforementioned spatial boundary conditions are applied. Even though there are different boundary conditions, and these often depend on the type of problem being studied, the most common types of boundary conditions are summarized below.

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10.5.1 Velocity Boundary Conditions

a) Prescribed velocity

In certain parts $\Gamma_{\bar{\mathrm{v}}}$ of the boundary of the control volume V being analyzed, the velocities are known (see Figure 10.13).

$$\boxed{\mathbf{v}(\mathbf{x},t) = \bar{\mathbf{v}}(\mathbf{x},t) \qquad \forall \mathbf{x} \in \Gamma_{\mathrm{v}}} \tag{10.98}$$

b) Impermeability condition

Usually, part of the boundary of the control volume V is composed of impermeable walls, Γ_{v_n}, which are assumed to be impervious to fluid, that is, they

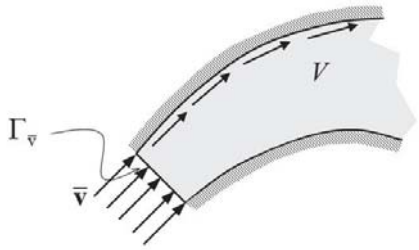

Figure 10.13: Velocity boundary conditions: prescribed velocity.

cannot be penetrated by said fluid. The mathematical expression of this condition is denoted as *impermeability condition* and it establishes that the relative velocity of the fluid, $\mathbf{v}_r$, with respect to the impermeable wall (assumed mobile and with a velocity $\mathbf{v}^*$) *in the direction normal to the boundary* must be null (see Figure10.14),

$$\mathrm{v}_n(\mathbf{x},t) = \underbrace{\mathbf{v}\cdot\mathbf{n}}_{\text{fluid}} = \underbrace{\mathbf{v}^*\cdot\mathbf{n}}_{\text{wall}} \qquad \forall \mathbf{x}\in\Gamma_{\mathrm{v}_n} \qquad \Longrightarrow$$
$$\boxed{\mathbf{v}_r\cdot\mathbf{n} = (\mathbf{v}-\mathbf{v}^*)\cdot\mathbf{n} = 0 \qquad \forall \mathbf{x}\in\Gamma_{\mathrm{v}_n}} \; . \tag{10.99}$$

In the particular case of a fixed boundary, this condition is reduced to $(\mathbf{v}^* = \mathbf{0}) \;\Rightarrow\; \mathbf{v}\cdot\mathbf{n} = 0 \quad \forall \mathbf{x}\in\Gamma_{\mathrm{v}_n}$.

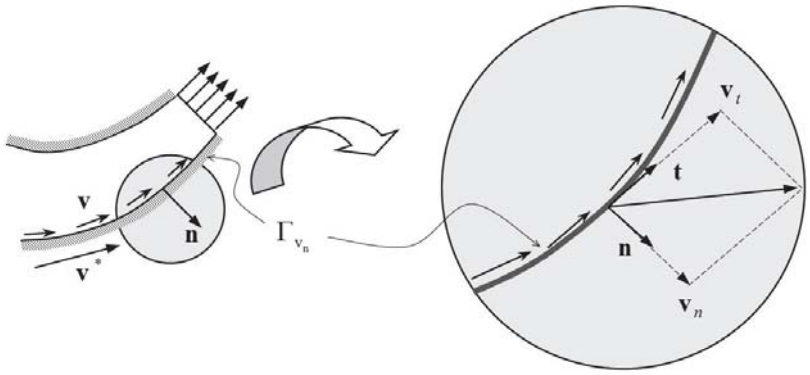

Figure 10.14: Velocity boundary conditions: impermeability condition.

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Remark 10.6. The impermeability condition is usually applied for *perfect fluids* (fluids without viscosity) in which the tangential component of the relative velocity between the fluid and the wall $\mathbf{v}_t$ (see Figure 10.14) is assumed to be non-null.

c) Adherence condition

In *viscous fluids* in contact with an impermeable wall, due to the effect of viscosity, the fluid is assumed to *adhere* to the wall (see Figure 10.15) and, thus, the relative velocity between the fluid and the wall $\mathbf{v}_r$ is null.

$$\mathbf{v}_r(\mathbf{x},t) = \mathbf{v} - \mathbf{v}^* = \mathbf{0} \qquad \forall \mathbf{x} \in \Gamma_{\mathrm{v}} \implies \boxed{\mathbf{v} = \mathbf{v}^* \quad \forall \mathbf{x} \in \Gamma_{\mathrm{v}}} \tag{10.100}$$

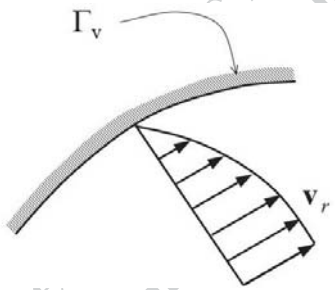

Figure 10.15: Velocity boundary conditions: adherence condition.

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10.5.2 Pressure Boundary Conditions

In certain parts Γ_σ of the boundary, the traction vector $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ can be prescribed (see Figure 10.16).

$$\boxed{\mathbf{t}(\mathbf{x},t) = \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}^*(\mathbf{x},t) \qquad \forall \mathbf{x} \in \Gamma_\sigma} \tag{10.101}$$

Under certain circumstances, only a part of the traction vector such as the *thermodynamic pressure* is prescribed. In effect, for a Newtonian fluid,

$$\begin{aligned} \boldsymbol{\sigma} &= -p\mathbf{1} + \lambda \operatorname{Tr}(\mathbf{d})\mathbf{1} + 2\mu\,\mathbf{d} \implies \\ \mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} &= -p\,\mathbf{n} + \lambda \operatorname{Tr}(\mathbf{d})\mathbf{n} + 2\mu\,\mathbf{d} \cdot \mathbf{n}\,, \end{aligned} \tag{10.102}$$

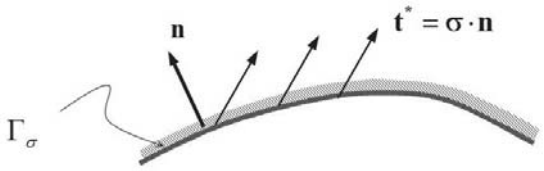

Figure 10.16: Pressure boundary conditions: prescribed traction vector.

which exposes how the thermodynamic pressure p is a part of the normal component of the traction vector $\mathbf{t}$. The prescription of the thermodynamic pressure on a part of the boundary Γ_p is written as

$$p(\mathbf{x},t) = p^*(\mathbf{x},t) \qquad \forall \mathbf{x} \in \Gamma_p \quad . \tag{10.103}$$

10.5.3 Mixed Boundary Conditions

In certain cases (such as the entrance and exit sections of pipes) the pressure (a part of the *normal* component of the traction vector) and the *tangential* components of the velocity (which are assumed to be null, see Figure 10.17) are prescribed.

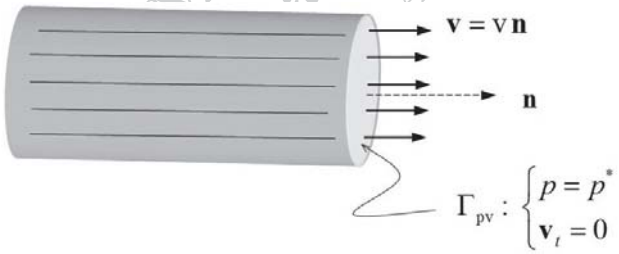

Figure 10.17: Mixed boundary conditions.

10.5.4 Boundary Conditions on Free Surfaces

Definition 10.5. A *free surface* is a contact surface between the air (atmosphere) and a fluid (generally water).

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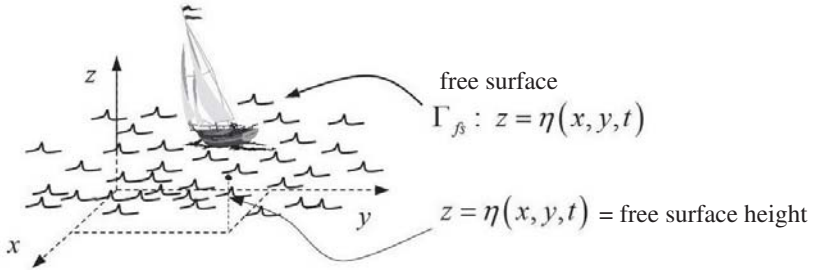

Figure 10.18: Free surface of the sea.

Examples of free surface[9] are the surface of the sea (see Figure 10.18) or the surface that separates the saturated and unsaturated parts of an embankment dam (see Figure 10.19).

A hypothesis with a clear physical sense that is frequently used in relation to a free surface is that such a surface is *a material surface* (constituted always by the same particles). This hypothesis implicitly establishes certain boundary conditions on the velocity field in the material surface Γ_{fs}. In effect, considering the free surface in Figure 10.18,

$$\Gamma_{fs} := \{\mathbf{x} \mid \phi(x,y,z,t) \equiv z - \eta(x,y,t) = 0\} \,, \tag{10.104}$$

and imposing the material character of the free surface (null material derivative, see Section 1.11 in Chapter 1),

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \mathbf{v}\cdot\nabla\phi = -\frac{\partial \eta}{\partial t} - \mathrm{v}_x \frac{\partial \eta}{\partial x} - \mathrm{v}_y \frac{\partial \eta}{\partial y} + \mathrm{v}_z \underbrace{\frac{\partial \phi}{\partial z}}_{=1} = 0 \,, \tag{10.105}$$

$$\boxed{\mathrm{v}_z(\mathbf{x},t) = \frac{\partial \eta}{\partial t} + \mathrm{v}_x \frac{\partial \eta}{\partial x} + \mathrm{v}_y \frac{\partial \eta}{\partial y} \qquad \forall \mathbf{x} \in \Gamma_{fs}} \,. \tag{10.106}$$

This condition establishes the dependency of the vertical component of the velocity v_z on the other components v_x and v_y.

Another boundary condition frequently imposed on free surfaces is that, in these surfaces, the thermodynamic pressure is known and equal to the atmospheric pressure[10],

$$\boxed{p(\mathbf{x},t) = P_{atm} \qquad \forall \mathbf{x} \in \Gamma_{fs}} \tag{10.107}$$

[9] In general, in fluid mechanics problems in which free surfaces appear, the position of these surfaces is not known and their geometrical characteristics become an unknown of the problem.

[10] The value of the atmospheric pressure is generally neglected ($P_{atm} \approx 0$).

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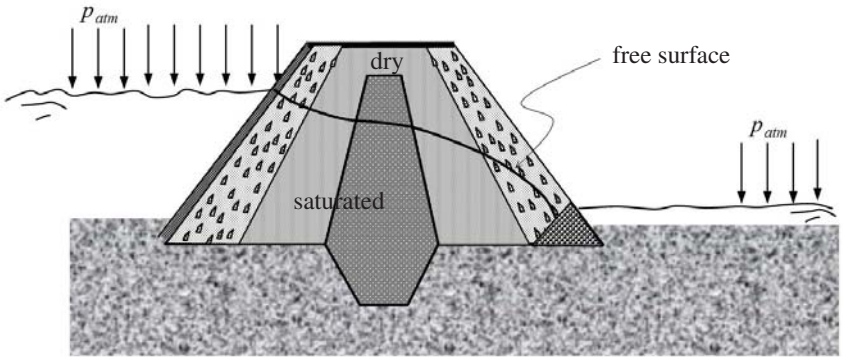

Figure 10.19: Free surface of an embankment dam.

Equation (10.107) allows identifying, in certain cases, the position of the free surface (once the pressure field is known) as the locus of points in the fluid in which the pressure is equal to the atmospheric pressure.

Equation of the free surface	$\Gamma_{fs} := \{\mathbf{x} \mid p(\mathbf{x},t) - P_{atm} = 0\}$	(10.108)

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10.6 Laminar and Turbulent Flows

10.6.1 Laminar Flow

The equations governing a fluid mechanics problem, described in the previous sections, are valid for a certain range of motion of the fluids, named *laminar flow* (or regime). Basically, laminar flow is physically characterized by the fact that the fluid moves in parallel layers that do not mix (see Figure 10.20).

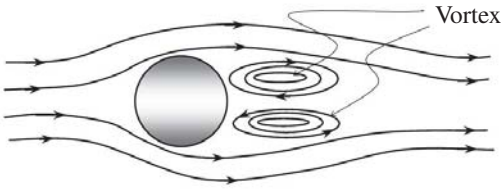

Figure 10.20: Laminar flow around an obstacle.

The character of a laminar flow is identified by the Reynolds number R_e

$$\boxed{\begin{array}{c}\text{Reynolds number: } R_e \stackrel{def}{=} \dfrac{V \times L}{\nu} \\ \begin{cases} V = \text{characteristic velocity of the fluid} \\ L = \text{characteristic length of the domain} \\ \nu = \text{kinematic viscosity } (\nu = \mu/\rho) \end{cases}\end{array}} \tag{10.109}$$

such that small values of the Reynolds number characterize laminar flows.

10.6.2 Turbulent Flow

When the velocity increases and the viscosity decreases, the Reynolds number (10.109) increases. For increasing values of this number, the initially laminar flow is seen to distort and become highly unstable. The flow can then be understood as being in a situation in which the velocity $\mathbf{v}(\mathbf{x},t)$, at a given point in space, randomly and rapidly fluctuates along time about a mean value $\bar{\mathbf{v}}(\mathbf{x},t)$ (see Figure 10.21). This situation is defined as *turbulent flow* (or regime).

Even though the equations of the fluid mechanics problem in general, and the Navier-Stokes equation in particular, are still valid in turbulent regime, certain circumstances (such as the difficulty in treating the mathematical problem and the impossibility of experimentally characterizing the rapid fluctuations of the variables of this problem) impose a singular treatment for turbulent flow. The mathematical characterization of turbulent regime is done, then, by means of the so-called *turbulence models*. These models are based on isolating the mean values of the velocity and pressure fields from their fluctuations and, then, the governing equations of the problem are obtained in terms of these mean values.

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Theory and Problems
Oliver and Agelet de Saracibar

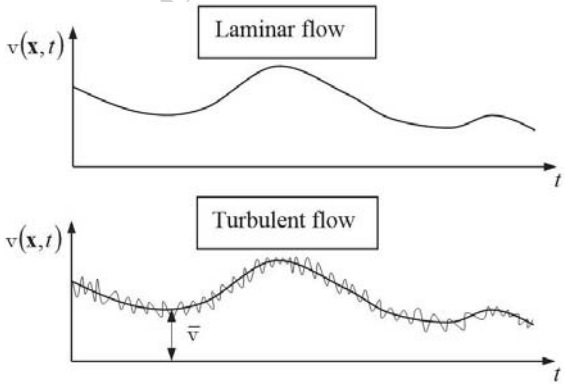

Figure 10.21: Variation of the velocity along time in laminar and turbulent flows.

10.7 Fluid Mechanics Formulas

10.7.1 Stress tensor for Newtonian fluids

(incompressible fluid, $\nabla \cdot \mathbf{v} = 0$)

Cartesian coordinates

$$\begin{aligned}
\sigma_x &= 2\mu \frac{\partial \mathrm{v}_x}{\partial x} - p \qquad & \tau_{xy} &= \tau_{yx} = \mu \left(\frac{\partial \mathrm{v}_x}{\partial y} + \frac{\partial \mathrm{v}_y}{\partial x} \right) \\
\sigma_y &= 2\mu \frac{\partial \mathrm{v}_y}{\partial y} - p \qquad & \tau_{yz} &= \tau_{zy} = \mu \left(\frac{\partial \mathrm{v}_y}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial y} \right) \\
\sigma_z &= 2\mu \frac{\partial \mathrm{v}_z}{\partial z} - p \qquad & \tau_{zx} &= \tau_{xz} = \mu \left(\frac{\partial \mathrm{v}_z}{\partial x} + \frac{\partial \mathrm{v}_x}{\partial z} \right)
\end{aligned} \tag{10.110}$$

Cylindrical coordinates

$$\begin{aligned}
\sigma_r &= 2\mu \frac{\partial \mathrm{v}_r}{\partial r} - p \qquad & \tau_{r\theta} &= \tau_{\theta r} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{\mathrm{v}_\theta}{r} \right) + \frac{1}{r} \frac{\partial \mathrm{v}_r}{\partial \theta} \right) \\
\sigma_\theta &= 2\mu \left(\frac{1}{r} \frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_r}{r} \right) - p \qquad & \tau_{\theta z} &= \tau_{z\theta} = \mu \left(\frac{\partial \mathrm{v}_\theta}{\partial z} + \frac{1}{r} \frac{\partial \mathrm{v}_z}{\partial \theta} \right) \\
\sigma_z &= 2\mu \frac{\partial \mathrm{v}_z}{\partial z} - p \qquad & \tau_{zr} &= \tau_{rz} = \mu \left(\frac{\partial \mathrm{v}_z}{\partial r} + \frac{\partial \mathrm{v}_r}{\partial z} \right)
\end{aligned} \tag{10.111}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} (r \mathrm{v}_r) + \frac{1}{r} \frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\partial \mathrm{v}_z}{\partial z} \tag{10.112}$$

Spherical coordinates

$$\begin{aligned}
\sigma_r &= 2\mu \frac{\partial \mathrm{v}_r}{\partial r} - p \\
\sigma_\theta &= 2\mu \left(\frac{1}{r} \frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_r}{r} \right) - p \\
\sigma_\phi &= 2\mu \left(\frac{1}{r \sin\theta} \frac{\partial \mathrm{v}_\phi}{\partial \phi} + \frac{\mathrm{v}_r}{r} + \frac{\mathrm{v}_\theta \cot\theta}{r} \right) - p
\end{aligned} \tag{10.113}$$

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$$\tau_{r\theta} = \tau_{\theta r} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{\mathrm{v}_\theta}{r} \right) + \frac{1}{r} \frac{\partial \mathrm{v}_r}{\partial \theta} \right)$$
$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu \left(\frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\mathrm{v}_\phi}{\sin\theta} \right) + \frac{1}{r \sin\theta} \frac{\partial \mathrm{v}_\theta}{\partial \phi} \right) \qquad (10.113 \text{ (cont.)})$$
$$\tau_{\phi r} = \tau_{r\phi} = \mu \left(\frac{1}{r \sin\theta} \frac{\partial \mathrm{v}_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{\mathrm{v}_\phi}{r} \right) \right)$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \mathrm{v}_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\mathrm{v}_\theta \sin\theta \right) + \frac{1}{r \sin\theta} \frac{\partial \mathrm{v}_\phi}{\partial \phi} \qquad (10.114)$$

10.7.2 Continuity Equation

Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left(\rho \mathrm{v}_x \right) + \frac{\partial}{\partial y} \left(\rho \mathrm{v}_y \right) + \frac{\partial}{\partial z} \left(\rho \mathrm{v}_z \right) = 0 \qquad (10.115)$$

Cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left(\rho r \mathrm{v}_r \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\rho \mathrm{v}_\theta \right) + \frac{\partial}{\partial z} \left(\rho \mathrm{v}_z \right) = 0 \qquad (10.116)$$

Spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(\rho r^2 \mathrm{v}_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\rho \mathrm{v}_\theta \sin\theta \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \left(\rho \mathrm{v}_\phi \right) = 0 \qquad (10.117)$$

10.7.3 Navier-Stokes Equation

(incompressible fluid, $\nabla \cdot \mathbf{v} = 0$; ρ and μ const.)

Cartesian coordinates

$$-\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 \mathrm{v}_x}{\partial x^2} + \frac{\partial^2 \mathrm{v}_x}{\partial y^2} + \frac{\partial^2 \mathrm{v}_x}{\partial z^2} \right) + \rho b_x = \rho \left(\frac{\partial \mathrm{v}_x}{\partial t} + \mathrm{v}_x \frac{\partial \mathrm{v}_x}{\partial x} + \mathrm{v}_y \frac{\partial \mathrm{v}_x}{\partial y} + v_z \frac{\partial \mathrm{v}_x}{\partial z} \right)$$
$$-\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 \mathrm{v}_y}{\partial x^2} + \frac{\partial^2 \mathrm{v}_y}{\partial y^2} + \frac{\partial^2 \mathrm{v}_y}{\partial z^2} \right) + \rho b_y = \rho \left(\frac{\partial \mathrm{v}_y}{\partial t} + \mathrm{v}_x \frac{\partial \mathrm{v}_y}{\partial x} + \mathrm{v}_y \frac{\partial \mathrm{v}_y}{\partial y} + \mathrm{v}_z \frac{\partial \mathrm{v}_y}{\partial z} \right)$$
$$-\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 \mathrm{v}_z}{\partial x^2} + \frac{\partial^2 \mathrm{v}_z}{\partial y^2} + \frac{\partial^2 \mathrm{v}_z}{\partial z^2} \right) + \rho b_z = \rho \left(\frac{\partial \mathrm{v}_z}{\partial t} + \mathrm{v}_x \frac{\partial \mathrm{v}_z}{\partial x} + \mathrm{v}_y \frac{\partial \mathrm{v}_z}{\partial y} + \mathrm{v}_z \frac{\partial \mathrm{v}_z}{\partial z} \right) \qquad (10.118)$$

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Cylindrical coordinates

$$-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\mathrm{v}_r\right)\right)+\frac{1}{r^2}\frac{\partial^2\mathrm{v}_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial\mathrm{v}_\theta}{\partial\theta}+\frac{\partial^2\mathrm{v}_r}{\partial z^2}\right)+\rho b_r=$$
$$=\rho\left(\frac{\partial\mathrm{v}_r}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_r}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_r}{\partial\theta}-\frac{\mathrm{v}_\theta^2}{r}+\mathrm{v}_z\frac{\partial\mathrm{v}_r}{\partial z}\right)$$

$$-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\mathrm{v}_\theta\right)\right)+\frac{1}{r^2}\frac{\partial^2\mathrm{v}_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial\mathrm{v}_r}{\partial\theta}+\frac{\partial^2\mathrm{v}_\theta}{\partial z^2}\right)+\rho b_\theta=$$
$$=\rho\left(\frac{\partial\mathrm{v}_\theta}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_\theta}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_\theta}{\partial\theta}+\frac{\mathrm{v}_r\mathrm{v}_\theta}{r}+\mathrm{v}_z\frac{\partial\mathrm{v}_\theta}{\partial z}\right)$$

$$-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mathrm{v}_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\mathrm{v}_z}{\partial\theta^2}+\frac{\partial^2\mathrm{v}_z}{\partial z^2}\right)+\rho b_z=$$
$$=\rho\left(\frac{\partial\mathrm{v}_z}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_z}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_z}{\partial\theta}+\mathrm{v}_z\frac{\partial\mathrm{v}_z}{\partial z}\right)$$

(10.119)

Spherical coordinates

$$-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mathrm{v}_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathrm{v}_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathrm{v}_r}{\partial\phi^2}+\right.$$
$$\left.-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\mathrm{v}_\theta\sin\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial\mathrm{v}_\phi}{\partial\phi}\right)+\rho b_r=$$
$$=\rho\left(\frac{\partial\mathrm{v}_r}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_r}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_r}{\partial\theta}+\frac{\mathrm{v}_\phi}{r\sin\theta}\frac{\partial\mathrm{v}_r}{\partial\phi}-\frac{\mathrm{v}_\theta^2+\mathrm{v}_\phi^2}{r}\right)$$

$$-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\mathrm{v}_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\mathrm{v}_\theta\sin\theta\right)\right)+\right.$$
$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathrm{v}_\theta}{\partial\phi^2}+\frac{2}{r^2}\frac{\partial\mathrm{v}_r}{\partial\theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial\mathrm{v}_\phi}{\partial\phi}\right)+\rho b_\theta=$$
$$=\rho\left(\frac{\partial\mathrm{v}_\theta}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_\theta}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_\theta}{\partial\theta}+\frac{\mathrm{v}_\phi}{r\sin\theta}\frac{\partial\mathrm{v}_\theta}{\partial\phi}+\frac{\mathrm{v}_r\mathrm{v}_\theta}{r}-\frac{\mathrm{v}_\phi^2\cot\theta}{r}\right)$$

(10.120)

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$$-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \mathrm{v}_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\mathrm{v}_\phi\sin\theta\right)\right)+\right.$$
$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 \mathrm{v}_\phi}{\partial \phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial \mathrm{v}_r}{\partial \phi}+\frac{2\cot\,\theta}{r^2\sin\theta}\frac{\partial \mathrm{v}_\theta}{\partial \phi}\right)+\rho b_\varphi=$$
$$=\rho\left(\frac{\partial \mathrm{v}_\phi}{\partial t}+\mathrm{v}_r\frac{\partial \mathrm{v}_\phi}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial \mathrm{v}_\phi}{\partial \theta}+\frac{\mathrm{v}_\phi}{r\sin\theta}\frac{\partial \mathrm{v}_\phi}{\partial \phi}+\frac{\mathrm{v}_\phi \mathrm{v}_r}{r}+\frac{\mathrm{v}_\theta \mathrm{v}_\phi}{r}\cot\theta\right) \quad (10.120\ \text{(cont.)})$$

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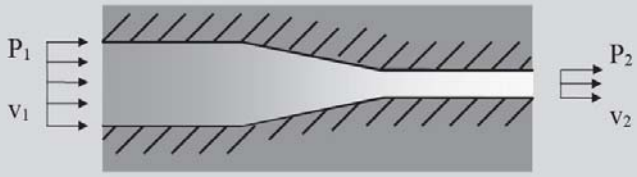
PROBLEMS

Problem 10.1 – *The barotropic fluid flowing inside the pipe shown in the figure below has the following kinetic equation of state.*

$$p = \beta \ln\left(\frac{\rho}{\rho_0}\right) \qquad (\beta \text{ and } \rho_0 \text{ const.})$$

Determine, for a steady-state regime, the exit pressure P_2 *in terms of the other variables shown in the figure.*



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Solution

The global spatial form of the *mass continuity equation* (5.22) states

$$\frac{d}{dt}\int_V \rho \, dV = 0 \, .$$

Using the *Reynolds Transport Theorem* (5.37) on this expression results in

$$\frac{d}{dt}\int_V \rho \, dV = \frac{\partial}{\partial t}\int_V \rho \, dV + \int_{\partial V} \rho \mathbf{v}\cdot\mathbf{n} \, dS \quad \Longrightarrow \quad \frac{\partial}{\partial t}\int_V \rho \, dV + \int_{\partial V} \rho \mathbf{v}\cdot\mathbf{n} \, dS = 0 \, ,$$

and introducing the conditions associated with a steady-state regime yields

$$\frac{\partial}{\partial t}\int_V \rho \, dV = 0 \quad \Longrightarrow \quad \int_{\partial V} \rho \mathbf{v}\cdot\mathbf{n} \, dS = 0 \, .$$

Applying this last expression to the problem described in the statement produces

$$-\rho_1 \mathrm{v}_1 S_1 + \rho_2 \mathrm{v}_2 S_2 = 0 \quad \Longrightarrow \quad \rho_1 \mathrm{v}_1 S_1 = \rho_2 \mathrm{v}_2 S_2 \, .$$

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Finally, isolating the density from the given *kinetic equation of state*,

$$p = \beta \ln\left(\frac{\rho}{\rho_0}\right) \qquad \Longrightarrow \qquad \rho = \rho_0 \, \mathrm{e}^{p/\beta} \, ,$$

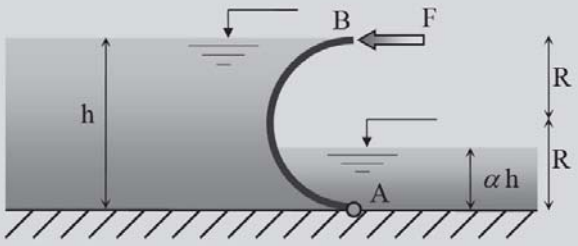
and introducing it into the previous one produces

$$\rho_0 \, \mathrm{e}^{\mathrm{P}_1/\beta} \, \mathrm{v}_1 \, S_1 = \rho_0 \, \mathrm{e}^{\mathrm{P}_2/\beta} \, \mathrm{v}_2 \, S_2 \quad \Longrightarrow \quad \mathrm{e}^{(\mathrm{P}_2 - \mathrm{P}_1)/\beta} = \frac{\mathrm{v}_1 \, S_1}{\mathrm{v}_2 \, S_2} \quad \Longrightarrow$$

$$\boxed{\mathrm{P}_2 = \mathrm{P}_1 + \beta \ln\left(\frac{\mathrm{v}_1 \, S_1}{\mathrm{v}_2 \, S_2}\right)} \, .$$

Engineers

Problem 10.2 – *Determine the value per unit of thickness of the horizontal force* F *that must be applied on point* B *of the semicircular floodgate shown in the figure such that the straight line* AB *remains vertical. The floodgate can rotate around the hinge* A *and separates two different height levels,* h *and* αh*, of a same fluid.*



HYPOTHESES:

1) The weight of the floodgate can be neglected.
2) The atmospheric pressure is negligible.

Solution

The only forces acting on the floodgate are the pressure forces of the fluids, the force F and the reaction in A (horizontal component H and vertical component V).

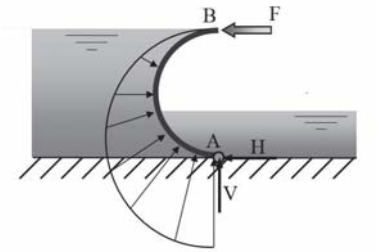


Figure A

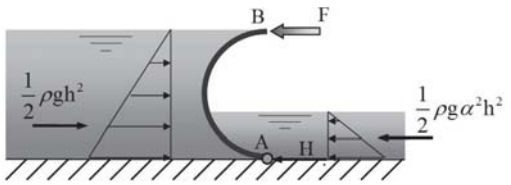


Figure B

Since the pressure exerted by the fluids is perpendicular to the surface of the floodgate and the floodgate is circular, the resultant force obtained by integrating the pressures on the surface are applied at the center of the circumference defined by the floodgate. Thus, posing the equilibrium of momentum with respect to the center of this circumference (see Figure A) results in

$$FR = HR \quad \Longrightarrow \quad H = F \, .$$

Imposing now the equilibrium of horizontal forces, knowing that the fluids exert a horizontal pressure with a triangular distribution (see Figure B), yields

$$2F + \frac{1}{2}(\rho g \alpha h)(\alpha h) = \frac{1}{2}(\rho g h)h \quad \Longrightarrow \quad \boxed{F = \frac{1}{4}\rho g h^2 \left(1 - \alpha^2\right)} \, .$$

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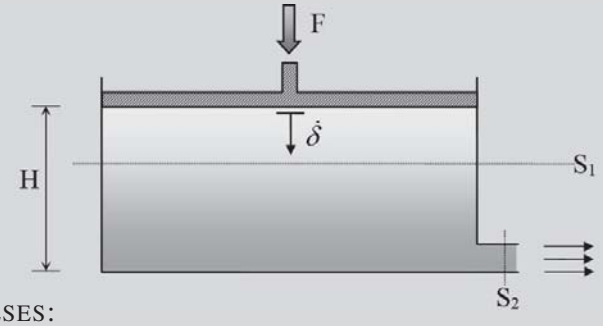
Problem 10.3 – *Determine the relation between the force* F *applied on the piston shown in the figure and its velocity of descent* $\dot{\delta}$.



HYPOTHESES:

1) Assume the fluid is an incompressible perfect fluid in steady-state regime.
2) The atmospheric pressure is negligible.
3) S_1 *and* S_2 *are the cross-sections.*
4) The density of the fluid is ρ.

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Solution

The stress state of a perfect fluid is known to be of the form $\boldsymbol{\sigma} = -p\mathbf{1}$ (see Section 9.3 in Chapter 9). The mass continuity equation (5.22) is applied to obtain the relation between the velocities of the fluid,

$$\mathrm{v}_1 \mathrm{S}_1 = \mathrm{v}_2 \mathrm{S}_2 \quad \Longrightarrow \quad \mathrm{v}_2 = \frac{\mathrm{S}_1}{\mathrm{S}_2}\mathrm{v}_1 = \frac{\mathrm{S}_1}{\mathrm{S}_2}\dot{\delta} \, .$$

Taking into account *Bernoulli's theorem* (10.63) between an arbitrary point in contact with the piston and another at the exit cross-section, both belonging to a same streamline, results in

$$\mathrm{H} + \frac{p}{\rho g} + \frac{\dot{\delta}^2}{2g} = 0 + 0 + \left(\frac{\mathrm{S}_1}{\mathrm{S}_2}\dot{\delta}\right)^2 \frac{1}{2g} \implies p = \frac{\rho}{2}\left(\left(\frac{\mathrm{S}_1}{\mathrm{S}_2}\right)^2 - 1\right)\dot{\delta}^2 - \rho g \mathrm{H}$$

Therefore, p must be constant for any point in contact with the piston ($\mathrm{x} = \mathrm{H}$). Then,

$$p = const. \qquad \forall \mathrm{x} = \mathrm{H} \quad \Longrightarrow \quad \mathrm{F} = p\,\mathrm{S}_1 \, .$$

Finally, the force F is related to $\dot{\delta}$ in the following manner.

$$\boxed{\mathrm{F} = \frac{\rho}{2}\mathrm{S}_1\left(\left(\frac{\mathrm{S}_1}{\mathrm{S}_2}\right)^2 - 1\right)\dot{\delta}^2 - \rho \mathrm{g} \mathrm{H} \mathrm{S}_1}$$

Continuum Mechanics for Engineers
Theory and Problems
X. Oliver and C. Agelet de Saracibar

Problem 10.4 – *A shear force* f^* *per unit of surface acts on an rigid plate of indefinite size with density* ρ^* *and thickness* t. *The plate slides at a velocity* v^* *in the longitudinal direction on a plane inclined at an angle* α *with respect to the horizontal longitudinal direction. Between the plate and the inclined plane are two distinct and immiscible Newtonian fluids with viscosities* μ_1 *and* μ_2, *which are distributed into two layers with the same thickness* h.

a) Establish the generic form of the pressure and velocity fields and argue the hypotheses used to determine them.

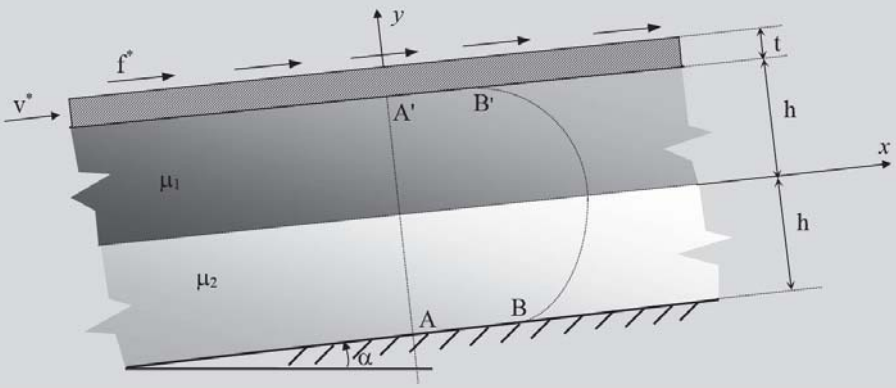
b) Integrate the corresponding differential equations and obtain, except for the integration constants, the distribution of pressures and velocities in each fluid.

c) Indicate and justify the boundary conditions that must be applied to determine the above integration constants.

d) Completely determine the pressure and velocity fields as well as the stresses in each fluid. Plot the distribution of each variable (velocities, pressure and

stresses) on a cross-section such as $\mathrm{A}-\mathrm{A}'$*, indicating the most significant values.*

e) Obtain the value of v^* *in terms of* f^* *and the volume flow rate* q *that flows through a semicircular section such as* $\mathrm{B}-\mathrm{B}'$*.*



HYPOTHESES:

1) Both fluids are incompressible.
2) Assume a steady-state regime.
3) The body forces of the fluids can be neglected.
4) The atmospheric pressure is negligible.

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Solution

a) Note that the z-dimension, perpendicular to the plane of the paper, does not intervene in the problem. Thus, the pressure and velocity fields are reduced to

$$p = p\,(x,y) \quad \text{and} \quad \mathbf{v} = \mathbf{v}\,(x,y) \overset{not}{\equiv} \left[\mathrm{v}_x\,(x,y)\ ,\ 0\,,\ 0\right]^T\ .$$

In fact, v_x does not depend on x either since the velocity should be the same in all the cross-sections of the type $\mathrm{A}-\mathrm{A}'$. If this is not acknowledged *a priori*, the *mass continuity equation* (5.22) may be imposed, considering the incompressible nature of the fluids, as follows.

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \quad \Longrightarrow \quad \nabla \cdot \mathbf{v} = 0 \quad \Longrightarrow \quad \frac{\partial \mathrm{v}_x}{\partial x} + \frac{\partial \mathrm{v}_y}{\partial y} = 0\,, \ \text{but} \ \mathrm{v}_y = 0$$

$$\Longrightarrow \quad \frac{\partial \mathrm{v}_x}{\partial x} = 0 \quad \Longrightarrow \quad \mathrm{v}_x = \mathrm{v}_x\,(y) \quad \Longrightarrow \quad \mathbf{v} \overset{not}{\equiv} \left[\mathrm{v}_x\,(y)\ ,\ 0\,,\ 0\right]^T$$

Therefore, the pressure and velocity fields are

$$\boxed{p = p(x,y) \qquad \text{and} \qquad \mathbf{v} \overset{not}{\equiv} \left[\mathrm{v}_x(y)\,,\,0\,,\,0\right]^T}\,.$$

b) The components of the Navier-Stokes differential equation (10.85) in Cartesian coordinates must be integrated to obtain the expressions of $\mathbf{v}$ and p,

$$x-\text{component} \quad \Longrightarrow \quad 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 \mathrm{v}_x}{\partial y^2}\,,$$

$$y-\text{component} \quad \Longrightarrow \quad 0 = -\frac{\partial p}{\partial y} \quad \Rightarrow \quad p = p(x)\,.$$

The pressure p only depends on x and the component v_x of $\mathbf{v}$ only depends on y. Therefore, the partial derivatives in the equation for the x-component can be replaced by total derivatives. In this way, an equality of functions is obtained in which the pressure term depends solely on x whilst the velocity term depends exclusively on y. Consequently, these terms must be constant.

$$\left.\begin{array}{l} \dfrac{dp}{dx} = \mu \dfrac{d^2 \mathrm{v}_x}{dy^2} \\ f(x) = f(y) \end{array}\right\} \quad \Longrightarrow \quad \frac{dp}{dx} = \mu \frac{d^2 \mathrm{v}_x}{dy^2} = k = const.$$

$$\frac{dp}{dx} = k \quad \Longrightarrow \quad \boxed{p(x) = kx + \mathrm{A}}$$

$$\frac{d^2 \mathrm{v}_x}{dy^2} = \frac{k}{\mu} \quad \Longrightarrow \quad \boxed{\mathrm{v}_x(y) = \frac{k}{2\mu} y^2 + \mathrm{B}y + \mathrm{C}}$$

To determine the stresses, the constitutive equation in Cartesian coordinates of Table10.2 is used,

$$\left.\begin{array}{l} \sigma_x = \sigma_y = \sigma_z = -p(x) \\ \tau_{xy} = \tau_{yx} = \mu \dfrac{\partial \mathrm{v}_x(y)}{\partial y} \end{array}\right\} \quad \Longrightarrow$$

$$\boxed{\boldsymbol{\sigma}(x,y) \overset{not}{\equiv} \begin{bmatrix} -kx - \mathrm{A} & \mu\left(\dfrac{k}{\mu} y + \mathrm{B}\right) & 0 \\ \mu\left(\dfrac{k}{\mu} y + \mathrm{B}\right) & -kx - \mathrm{A} & 0 \\ 0 & 0 & -kx - \mathrm{A} \end{bmatrix}}$$

where the constants in these expressions (k, A, B, C) are different for each fluid.

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c) The boundary conditions that must be applied in this problem are:

VELOCITY BOUNDARY CONDITIONS

1. $\mathrm{v}_x^1(y)\big|_{y=\mathrm{h}} = \mathrm{v}^*$, since the plate moves at a velocity v^* and $\mu > 0$.
2. $\mathrm{v}_x^2(y)\big|_{y=-\mathrm{h}} = 0$, since the inclined plane does not move and $\mu > 0$.
3. $\mathrm{v}_x^1(y)\big|_{y=0} = \mathrm{v}_x^2(y)\big|_{y=0}$, which is the continuity condition for $\mathbf{v}$ at the interface between the two fluids.

PRESSURE BOUNDARY CONDITIONS

In the fluid with density μ_1, the pressure is prescribed for $y = \mathrm{h}$ or, directly, since p does not depend on y (because the weight of the fluid is neglected), the pressure p^1 is prescribed in the whole domain of this fluid. The value of p^1 corresponds to the pressure that the plate exerts on the fluid with density μ_1, which is the projection of the plate's weight in the direction of the y-axis.

4. $W = \rho^* g t$ is the weight of a section of the plate with unit length, according to the x- and z-axis. Here, $p_{atm} = 0$ has been considered.
5. $p^1 = \rho^* g t \cos\alpha$, $\forall \mathbf{x}$ is the projection on the y-axis. Since a unit length has been considered, the weight is directly the exerted pressure.
6. $p^1\big|_{y=0} = p^2\big|_{y=0}$ $\forall \mathbf{x}$ is the continuity condition for the pressure in the interface between the two fluids.

STRESS BOUNDARY CONDITIONS

The continuity condition for stresses that must be imposed in the interface between the two fluids does not affect the complete tensor $\boldsymbol{\sigma}$. Instead, only the traction vector $\mathbf{t}$ is affected. The condition

$$\mathbf{t}^1\big|_{y=0} = -\,\mathbf{t}^2\big|_{y=0}$$

must be satisfied. Considering that the unit normal vector $\mathbf{n}$ is the exterior normal vector, then

$$\mathbf{n}^1 \overset{not}{\equiv} [0\,,\,-1\,,\,0]^T \qquad \text{and} \qquad \mathbf{n}^2 \overset{not}{\equiv} [0\,,\,1\,,\,0]^T \ .$$

Hence, the shear stresses must satisfy:

7. $\tau_{xy}^1\big|_{y=0} = \tau_{xy}^2\big|_{y=0}$

d) Only 7 boundary conditions have been established and 8 constants must be determined, but since some equations include two constants, it suffices. Replacing the expressions of p, $\mathbf{v}$ and $\boldsymbol{\sigma}$ in the boundary conditions results in:

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$$\frac{k_1}{2\mu_1}\mathrm{h}^2+\mathrm{B}_1\mathrm{h}+\mathrm{C}_1=\mathrm{v}^*$$

$$\frac{k_2}{2\mu_2}\mathrm{h}^2-\mathrm{B}_2\mathrm{h}+\mathrm{C}_2=0$$

$$\mathrm{C}_1=\mathrm{C}_2$$

$$k_1 x+\mathrm{A}_1=\rho^* g t\cos\alpha\ ,\ \forall x\implies\begin{cases}k_1=0\\ \mathrm{A}_1=\rho^* g t\cos\alpha\end{cases}$$

$$k_1 x+\mathrm{A}_1=k_2 x+\mathrm{A}_2\ ,\ \forall x\implies\begin{cases}k_1=\mathrm{k}_2=0\\ \mathrm{A}_1=\mathrm{A}_2=\rho^* g t\cos\alpha\end{cases}$$

$$\left.\begin{array}{c}y=0\\ k_1=\mathrm{k}_2=0\end{array}\right\}\implies\ \mu_1\mathrm{B}_1=\mu_2\mathrm{B}_2$$

Solving and replacing these values in the expressions for the pressure, velocity and stress obtained in *b)* results in

$$\mathrm{v}_x^1(y)=\frac{\mathrm{v}^*}{1+\dfrac{\mu_1}{\mu_2}}\left(\frac{y}{\mathrm{h}}+\frac{\mu_1}{\mu_2}\right)$$

$$\mathrm{v}_x^2(y)=\frac{\mu_1}{\mu_2}\frac{\mathrm{v}^*}{1+\dfrac{\mu_1}{\mu_2}}\left(\frac{y}{\mathrm{h}}+1\right)$$

$$p^1=p^2=\rho^* g t\cos\alpha=const.$$

$$\tau_{xy}^1=\tau_{xy}^2=\mu_1\frac{\mathrm{v}^*}{\mathrm{h}\left(1+\dfrac{\mu_1}{\mu_2}\right)}=const.$$

e) To determine the relation between f^* and v^*, the equilibrium of forces on a unit element of the plate is posed. Three forces act on this element:

1) The force f^* that pushes the plate in the positive direction of the x-axis.
2) The projection of the plate's own weight in the direction of the x-axis. This force pulls the plate in the negative direction of the x-axis.
3) The shear force of the fluid on the plate, which resists the motion of the plate and, thus, acts in the negative direction of the x-axis.

To determine the sign criterion of this last force, the stresses acting on an element of the fluid domain are drawn:

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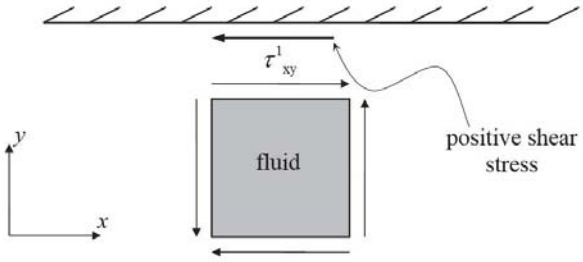

Posing the equilibrium of forces yields

$$\mathrm{f}^* = \rho^* g t \sin\alpha + \mu_1 \frac{\mathrm{v}^*}{\left(1+\dfrac{\mu_1}{\mu_2}\right)\mathrm{h}}$$

and, isolating v^*, produces the velocity in terms of the shear force,

$$\boxed{\mathrm{v}^* = \frac{\mathrm{h}}{\mu_1}\left(1+\frac{\mu_1}{\mu_2}\right)(\mathrm{f}^* - \rho^* g t \sin\alpha)} \,.$$

To compute the volume flow rate that flows across the surface $\mathrm{B}-\mathrm{B}'$, one must take into account that the fluids are incompressible and, thus, the volume flow rate crossing the curved surface is the same as if a straight segment joining B and B' was considered, that is,

$$\mathrm{q} = \int_{\mathrm{BB}'\,curved} \mathbf{v}\cdot\mathbf{n}\, dS = \int_{\mathrm{BB}'\,straight} \mathbf{v}\cdot\mathbf{n}\, dS = \int_{-\mathrm{h}}^{\mathrm{h}} \mathrm{v}_x(y)\; dy \,.$$

Then, replacing the expressions found in *d)* for the velocity v_x and integrating results in the volume flow

$$\boxed{\mathrm{q} = \mathrm{v}^* \mathrm{h}\left(\frac{1}{2}+\frac{\mu_1}{\mu_1+\mu_2}\right)} \,.$$

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Problem 10.5 – *Figure A shows the cross-section of a damper of indefinite length composed of a piston* ABA′B′ *that slides inside a container filled with an incompressible Newtonian fluid with viscosity* μ. *The piston descends at a velocity* $\dot{\delta}(t)$, *producing a lateral flow of fluid between the piston and the walls (see Figure B).*

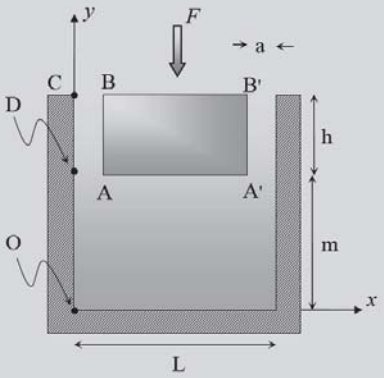

Figure A

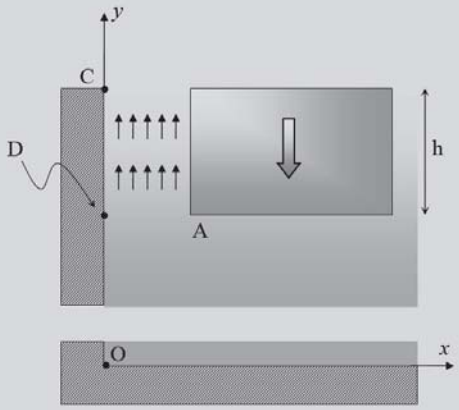

Figure B

a) *Determine the pressure and velocity fields in the zone of the fluid shown in Figure B (zone* ABCD*), except for the integration constants.*
b) *Indicate and justify the boundary conditions that must be applied to determine the above integration constants.*
c) *Completely determine the pressure and velocity fields in zone* ABCD *of the fluid.*
d) *Determine the expression of the stress tensor in zone* ABCD *of the fluid.*
e) *Assuming that the stress* σ_y *in the surface* $A-A'$ *is uniform and equal to the stress in point* A, *prove there exists a relation of the form* $F = \eta \dot{\delta}(t)$, *where* F *is the force per unit of length applied on the piston and* $\dot{\delta}(t)$ *is the velocity of descent of said piston. Compute the value of* η.

HYPOTHESES:

1) *The body forces of the fluid can be neglected.*
2) *The weight of the piston can be neglected.*
3) *Assume a steady-state regime.*
4) *The atmospheric pressure is negligible.*

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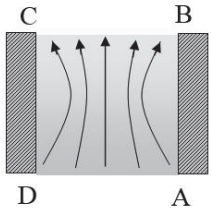
Solution

a) The problem is not defined in the z-direction, the direction perpendicular to the plane of the paper, and, thus, is independent of the z variable. Then, consider the bidimensional situation

$$\mathbf{v} \overset{not}{\equiv} \left[\mathrm{v}_x(x,y)\ ,\ \ \mathrm{v}_y(x,y)\ ,\ \ 0 \right]^T\ .$$

On the other hand, $\mathrm{v}_x = 0$ must be satisfied on the walls AB and CD, owing to the impermeability condition (a fluid cannot penetrate through a solid).
For convenience, an additional approximate hypothesis is introduced to further simplify the problem: it will be assumed that $\mathrm{v}_x = 0$ in all the zone ABCD of the fluid. However, the streamlines have, in fact, the approximate form shown in the figure to the right. It is even possible that vortexes are formed in this region if there is a high velocity.



In short, the velocity and pressure fields are assumed to be of the form

$$\mathbf{v} \overset{not}{\equiv} \left[0\ ,\ \ \mathrm{v}_y(x,y)\ ,\ \ 0 \right]^T \quad \text{and} \quad p = p(x,y)\ .$$

The *mass continuity equation* (5.22) for an incompressible fluid ($\rho = const.$) is reduced to $\nabla \cdot \mathbf{v} = 0$ and, for this particular problem,

$$\frac{\partial \mathrm{v}_y}{\partial y} = 0 \quad \Longrightarrow \quad \mathrm{v}_y = \mathrm{v}_y(x)\ .$$

Then, the velocity remains constant for a same vertical line since the spatial description of the velocity does not depend on y.
Now, the *Navier-Stokes equation* (10.85) in Cartesian coordinates is imposed, considering the hypotheses given in the statement of the problem and the additional assumptions described above. Since the problem is bidimensional, the z-component of the equation does not provide information.

$$\left.\begin{aligned} 0 = -\frac{\partial p}{\partial x} \quad &\Longrightarrow \quad p = p(y) \\ 0 = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 \mathrm{v}_y}{\partial x^2} \quad & \end{aligned}\right\} \quad \Longrightarrow \quad \frac{\partial p}{\partial y} = \mu \frac{\partial^2 \mathrm{v}_y}{\partial x^2}$$

The term in the right-hand side of the equation depends solely on x and the one in the left-hand side depends only on y, therefore both terms must be constant.

$$\frac{\partial p}{\partial y} = k \quad \Longrightarrow \quad \boxed{p = ky + C_1}$$

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$$\mu \frac{\partial^2 \mathrm{v}_y}{\partial x^2} = k \quad \Longrightarrow \quad \frac{\partial \mathrm{v}_y}{\partial x} = \frac{k}{\mu} x + C_2 \quad \Longrightarrow \quad \boxed{\mathrm{v}_y(x) = \frac{1}{2}\frac{k}{\mu} x^2 + C_2 x + C_3}$$

b) The boundary conditions that must be applied in this problem are:

VELOCITY BOUNDARY CONDITIONS

1. $\mathrm{v}_y(x)\Big|_{x=0} = 0\,, \ \forall y$, since there is no relative displacement of the fluid with respect to the wall.
2. $\mathrm{v}_y(x)\Big|_{x=\mathrm{a}} = -\dot{\delta}\,, \ \forall y$, again, since there is no relative displacement.

PRESSURE BOUNDARY CONDITIONS

3. $p(y)\Big|_{y=\mathrm{m+h}} = p_{atm} = 0$

VOLUME FLOW RATE BOUNDARY CONDITIONS

In an incompressible fluid the entrance and exit volume flow rates are the same, $Q_{in} = Q_{out}$, where

$$Q = \int_S \mathbf{v} \cdot \mathbf{n} \, dS \,.$$

The piston descends at a velocity $\dot{\delta}$ and, thus, its cross-section is introduced into the fluid, pushing it upwards. Then, the entrance volume flow rate can be defined as (velocity · surface),

$$Q_{in} = \dot{\delta} \cdot \mathrm{L} \,.$$

On the other hand, the exit volume flow rate, flowing in the space left between the piston and the lateral walls, is determined by means of the general expression for volume flow rate

$$Q_{out} = 2 \int_{S_\mathrm{a}} \mathbf{v} \cdot \mathbf{n} \, dS = 2 \int_0^{\mathrm{a}} \mathrm{v}_y(x) \ dx \,.$$

Finally, equating the entrance and exit volume flow rates results in:

4. $$2 \int_0^a \mathrm{v}_y(x)\, dx = \dot{\delta}\, \mathrm{L}$$

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c) The constants are determined by means of the boundary conditions described in *b)* as follows:

$$\mathrm{v}_y(x)\Big|_{x=0} = 0 \quad \Longrightarrow \quad \mathrm{v}_y(0) = C_3 \quad \Longrightarrow \quad \boxed{C_3 = 0}$$

$$\mathrm{v}_y(x)\Big|_{x=\mathrm{a}} = -\dot{\delta} \quad \Longrightarrow \quad \mathrm{v}_y(\mathrm{a}) = \frac{1}{2}\frac{k}{\mu}\mathrm{a}^2 + C_2\,\mathrm{a} \quad \Longrightarrow \quad C_2 = -\frac{\dot{\delta}}{\mathrm{a}} - \frac{k}{2\mu}\mathrm{a}$$

$$2\int_0^{\mathrm{a}} \mathrm{v}_y(x)\; dx = 2\int_0^{\mathrm{a}} \left(\frac{k}{2\mu}x^2 + C_2 x\right) dx = 2\left(\frac{k}{2\mu}\frac{\mathrm{a}^3}{3} + C_2\frac{\mathrm{a}^2}{2}\right) = \dot{\delta}\mathrm{L} \quad \Longrightarrow$$

$$\boxed{k = -\frac{6\mu}{\mathrm{a}^3}\dot{\delta}\,(\mathrm{a}+\mathrm{L})} \qquad \text{and} \qquad \boxed{C_2 = \frac{\dot{\delta}}{\mathrm{a}}\left(2 + 3\frac{\mathrm{L}}{\mathrm{a}}\right)}$$

$$p(y)\Big|_{y=\mathrm{m}+\mathrm{h}} = 0 \quad \Longrightarrow \quad k\,(\mathrm{m}+\mathrm{h}) + C_1 = 0 \quad \Longrightarrow \quad \boxed{C_1 = \frac{6\mu}{\mathrm{a}^3}\dot{\delta}\,(\mathrm{a}+\mathrm{L})\,(\mathrm{m}+\mathrm{h})}$$

Introducing these values in the expressions for the pressure and velocity obtained in *a)* results in:

$$\boxed{\begin{aligned} p = p(y) &= \frac{6\mu}{\mathrm{a}^3}\dot{\delta}\,(\mathrm{a}+\mathrm{L})\,(\mathrm{m}+\mathrm{h}-y) \\ \mathrm{v}_y(x) &= -\frac{3}{\mathrm{a}^3}(\mathrm{a}+\mathrm{L})\,\dot{\delta}x^2 + \frac{\dot{\delta}}{\mathrm{a}}\left(2+3\frac{\mathrm{L}}{\mathrm{a}}\right)x \end{aligned}}$$

d) The stresses in zone ABCD of the fluid are computed by means of the constitutive equation in Cartesian coordinates of Table 10.2. Using the expressions for the pressure and velocity fields obtained in *c)* yields

$$\boxed{\boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} -p & \mu\dfrac{\partial \mathrm{v}_y}{\partial x} & 0 \\ \mu\dfrac{\partial \mathrm{v}_y}{\partial x} & -p & 0 \\ 0 & 0 & -p \end{bmatrix}}$$

where

$$\mu\frac{\partial \mathrm{v}_y}{\partial x} = \mu\dot{\delta}\left(-6\frac{x}{\mathrm{a}^3}(\mathrm{a}+\mathrm{L}) + 3\frac{\mathrm{L}}{\mathrm{a}^2} + \frac{2}{\mathrm{a}}\right).$$

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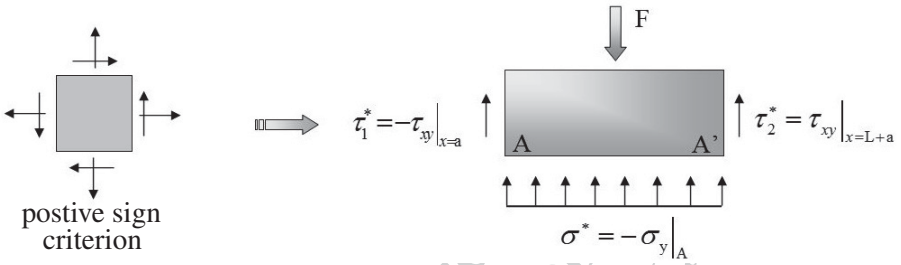
COMMENT

When the piston descends, the steady-state regime hypothesis is, in fact, not completely rigorous since, at some point, the piston will reach the lowest point of its trajectory and the flow will vary. To be able to apply this hypothesis, either $(\mathrm{m}+\mathrm{h})$ must be a very large length or $\dot{\delta}$ must be a very low velocity.

e) The stresses acting on the piston must be computed to obtain the resultant forces and, then, the equilibrium of forces is applied to determine the expression for F. These stresses are:



The stresses in the inferior surface of the piston are

$$\sigma^* = -\sigma_y\Big|_{\mathrm{A}} = p(y)\Big|_{y=\mathrm{m}} = k\,\mathrm{m} + C_1 \quad \Longrightarrow \quad \sigma^* = \frac{6\mu}{\mathrm{a}^3}(\mathrm{a}+\mathrm{L})\,\mathrm{h}\dot{\delta}\ .$$

In the lateral surfaces, due to symmetry, $\tau_1^* = \tau_2^*$ and, therefore, only τ_1^* needs to be computed,

$$\tau_1^* = \tau_2^* = -\tau_{xy}\Big|_{x=\mathrm{a}} = -\mu\dot{\delta}\left(-6\frac{\mathrm{a}}{\mathrm{a}^3}(\mathrm{a}+\mathrm{L}) + 3\frac{\mathrm{L}}{\mathrm{a}^2} + \frac{2}{\mathrm{a}}\right) \quad \Longrightarrow$$

$$\tau_1^* = \tau_2^* = \frac{\mu\dot{\delta}}{\mathrm{a}}\left(3\frac{\mathrm{L}}{\mathrm{a}} + 4\right)\ .$$

Imposing the equilibrium of forces (since $\dot{\delta}$ is a constant velocity),

$$\mathrm{F} = \mathrm{L}\sigma^* + \mathrm{h}\,\tau_1^* + \mathrm{h}\,\tau_2^* \quad \Longrightarrow$$

$$\boxed{\mathrm{F} = \eta\dot{\delta} \qquad \text{with} \quad \eta = \frac{2\mu\mathrm{h}}{\mathrm{a}}\left(3\frac{\mathrm{L}^2}{\mathrm{a}^2} + 6\frac{\mathrm{L}}{\mathrm{a}} + 4\right)}$$

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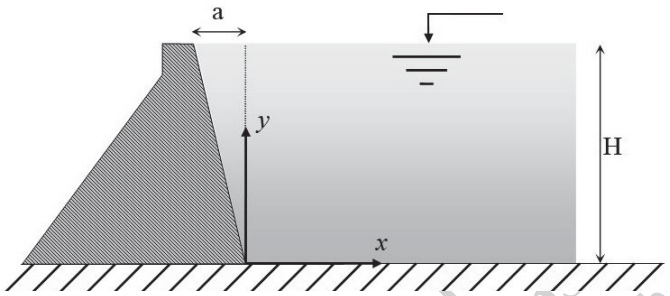
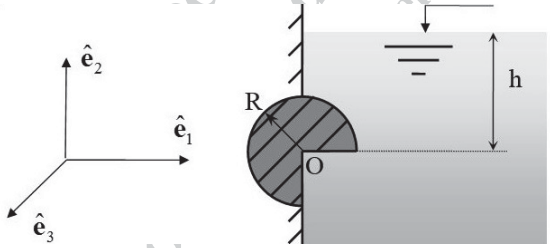
EXERCISES

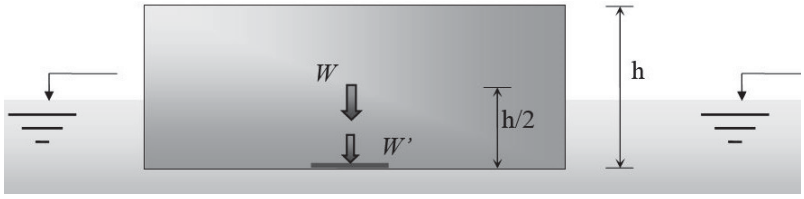
10.1 – *Compute the horizontal and vertical components of the resultant of the actions, per unit of length, exerted by the water on the gravity dam shown in the figure.*



10.2 – *The wall of a tank has a valve that rotates about point* O *as shown in the figure. Compute the resultant force and moment, per unit of thickness, that the fluid exerts on the valve. The weight of the valve can be neglected.*

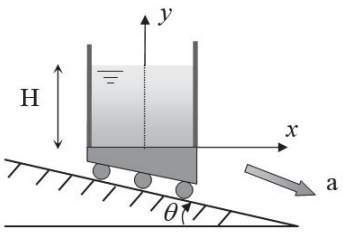


10.3 – *Determine the weight of the ballast* W′ *required at the bottom of the crate shown in the figure, whose weight is* W*, such that it is maintained afloat in stable equilibrium.*



NOTE*: The water has a density* ρ *and the weights are per unit of thickness.*

10.4 – *A container filled with water up to a height* H *is placed on an inclined plane with angle* θ *and dropped such that it slides down this plane with a constant acceleration value* a*. Determine the distribution of pressures and the equation of the free surface in terms of* a, H, θ *and the atmospheric pressure* p_a.



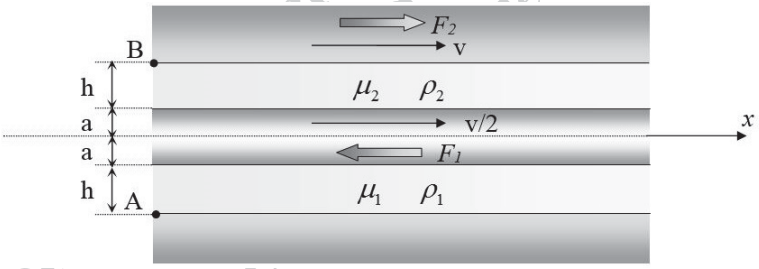
10.5 – *A plate of indefinite size and thickness* 2a *separates two incompressible Newtonian fluids that move between two rigid boundaries of indefinite length placed at a distance* h *from the plate, as shown in the figure. The plate and the top boundary move at velocities* v/2 *and* v*, respectively. Determine:*

a) The pressure, velocity and stress fields in terms of the integration constants.

b) The integration constants, by applying the adequate boundary conditions.

c) The forces per unit of surface F_1 *and* F_2 *exerted on the plate and the top boundary needed to produce the described motion.*

d) The dissipated energy, per unit of time and of surface perpendicular to the plane of the paper, due to viscous effects.



Additional hypotheses:

1) The pressures at points A *and* B *are* p_A *and* p_B*, respectively.*

2) Consider a steady-state regime.

3) Due to the indefinite character of the x-direction, the flow and its properties can be considered to be invariable in this direction.

10.6 – *A volume flow rate Q of an incompressible isotropic Newtonian fluid flows in steady-state regime between the plate and the horizontal surface shown in the figure. The plate is kept horizontal and immobile by means of a force with horizontal and vertical components* H *and* V*, respectively, acting on an appropriate point of said plate. Determine:*

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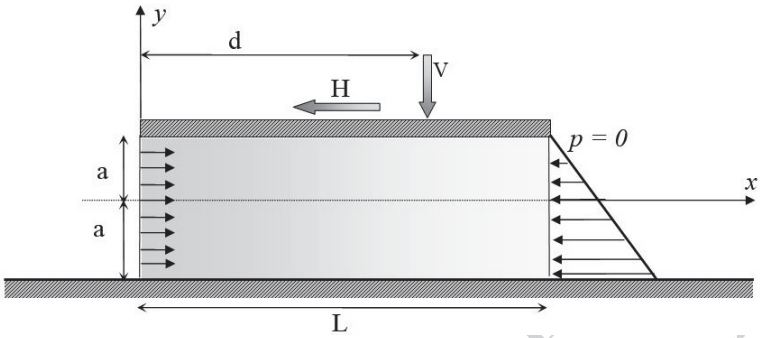
a) The pressure and velocity fields.

b) The value of the vertical component of the force and the distance d *from the origin to its application point, such that the plate does not rotate.*

c) The value of the horizontal component of the force.



Additional hypotheses:

1) The flow is assumed to be parallel to the x-y plane.
2) Inertial forces can be neglected.
3) The volume flow rate, Q, and the components of the force, V *and* H*, are considered per unit of length in the z-direction.*
4) The weight of the plate and the atmospheric pressure are negligible.

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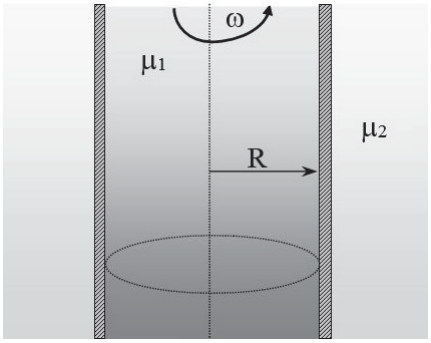
10.7 – *A cylindrical shell of indefinite length and internal radius* R *rotates in steady-state regime at an angular velocity* ω *inside an infinite domain occupied by an incompressible Newtonian fluid with viscosity* μ_2*. A different incompressible Newtonian fluid, with viscosity* μ_1*, is contained inside the cylindrical shell. Determine:*

a) The pressure and velocity fields of the internal fluid.

b) The pressure and velocity fields of the external fluid.

c) The moment that must be applied on the cylindrical shell to maintain its velocity.



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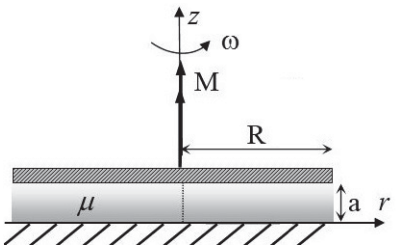
10.8 – *A disc of radius* R *rotates with a constant angular velocity* ω *at a distance* a *from a horizontal surface. Between the disc and the surface is an incompressible Newtonian fluid with viscosity* μ*. Determine:*

a) The velocity field of the fluid in terms of the integration constants.

b) The value of the integration constants, by applying the appropriate boundary conditions, and the complete expression of the velocity field.

c) The pressure field and the shear stress $\tau_{z\theta}$*.*

d) The value of the moment M *that must be applied on the axis of the disc to maintain the described motion.*



Additional hypotheses:

1) The rotation of the disc is sufficiently slow to neglect the inertial forces.
2) The effect of the lateral walls (fluid-wall friction effects) can be neglected.
3) The velocity field varies linearly with the distance to the inferior surface.
4) Assume a steady-state regime.

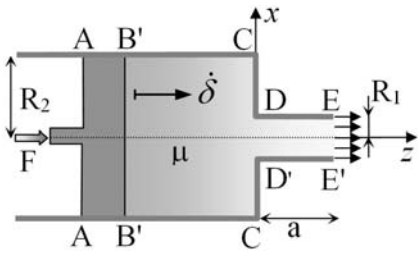
10.9 – *The cross-section of a cylindrical piston* ABA′B′ *that slides inside a container filled with an incompressible Newtonian fluid with viscosity* μ *is shown in the figure. The motion of the piston, at a velocity* $\dot{\delta}$*, causes the fluid to flow through the pipe* DED′E′*.*

a) Determine the pressure and velocity fields of the fluid in zone DED′E′ *in terms of the integration constants.*

b) Indicate and justify the boundary conditions that must be applied to determine the value of the integration constants. Determine these constants and the complete expressions of the pressure and velocity fields.

c) Compute the stresses in zone DED′E′ *of the fluid.*

d) Assuming that the stress normal to the surface BB′ *in the fluid is constant and equal to the pressure in points* D *and* D′*, prove there exists a relation between the force* F *applied on the piston and its velocity* $\dot{\delta}$*, and that said relation is of the form* $F = \eta \dot{\delta}$*. Determine the value of* η*.*



Additional hypotheses:

1) The body forces of the fluid and weight of the piston can be neglected.
2) Assume a steady-state regime.
3) The atmospheric pressure is negligible.

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