CH.2. DEFORMATION AND STRAIN
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2.1 Introduction

Ch.2. Deformation and Strain
Deformation

- **Deformation**: transformation of a body from a reference configuration to a current configuration.

- Focus on the relative movement of a given particle w.r.t. the particles in its neighbourhood (at differential level).

- It includes changes of size and shape.
2.2 Deformation Gradient Tensors

Ch.2. Deformation and Strain
Continuous Medium in Movement

$\Omega_0$: non-deformed (or reference) configuration, at reference time $t_0$.

$X$: Position vector of a particle at reference time.

$\Omega$ or $\Omega_t$: deformed (or present) configuration, at present time $t$.

$x$: Position vector of the same particle at present time.

$\varphi(X,t)$

Reference or non-deformed

Present or deformed
Fundamental Equation of Deformation

- The Equations of Motion:

\[
x_i = \varphi_i (X_1, X_2, X_3, t) = x_i (X_1, X_2, X_3, t) \quad i \in \{1, 2, 3\}
\]

\[
x = \varphi (X, t) = x(X, t)
\]

- Differentiating w.r.t. \( X \): 

\[
\begin{align*}
    dx_i &= \frac{\partial x_i (X, t)}{\partial X_j} dX_j = F_{ij} (X, t) dX_j \quad i, j \in \{1, 2, 3\} \\
    dx &= \frac{\partial x (X, t)}{\partial X} \cdot dX = F (X, t) \cdot dX
\end{align*}
\]
Material Deformation Gradient Tensor

- The (material) deformation gradient tensor:

\[
\begin{align*}
F(X,t) &= x(X,t) \otimes \nabla \\
F_{ij} &= \frac{\partial x_i}{\partial X_j} \quad i,j \in \{1,2,3\}
\end{align*}
\]

- **Remark:**

The material Nabla operator is defined as:

\[
\nabla = \begin{bmatrix}
\frac{\partial}{\partial X_1} \\
\frac{\partial}{\partial X_2} \\
\frac{\partial}{\partial X_3}
\end{bmatrix}
\]

- **F(X,t):**
  - is a primary measure of deformation
  - characterizes the variation of relative placements in the neighborhood of a **material** point (particle).

\[
dx = F(X,t) \cdot dX
\]
Inverse (spatial) Deformation Gradient Tensor

- The inverse Equations of Motion:

\[
\begin{align*}
X_i &= \varphi_i^{-1}(x_1, x_2, x_3, t) = X_i(x_1, x_2, x_3, t) \\
X &= \varphi^{-1}(x, t) = X(x, t)
\end{align*}
\]

- Differentiating w.r.t. \( x \):

\[
\begin{align*}
dX_i &= \frac{\partial X_i(X, t)}{\partial x_j} \, dx_j = \mathbf{F}^{-1}_{ij}(x, t) \, dx_j \\
\mathbf{dX} &= \frac{\partial X(x, t)}{\partial \mathbf{x}} \cdot d\mathbf{x} = \mathbf{F}^{-1}(x, t) \cdot d\mathbf{x}
\end{align*}
\]
Inverse (spatial) Deformation Gradient Tensor

- The *spatial* (or inverse) deformation gradient tensor:
  \[ F^{-1}(x, t) = X(x, t) \otimes \nabla \]
  \[ F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} \quad i, j \in \{1, 2, 3\} \]

- \( F^{-1}(x, t) \):
  - is a primary measure of deformation
  - characterizes the variation of relative placements in the neighbourhood of a spatial point.
  - It is *not* the spatial description of the material deformation gradient tensor.

**REMARK**

The spatial Nabla operator is defined as:

\[ \nabla \equiv \frac{\partial}{\partial x_i} \hat{e}_i \]

\[ [\nabla] = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} \]

\[ [F^{-1}] = [X \otimes \nabla] = \nabla^T = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \]
Properties of the Deformation Gradients

- The *spatial* deformation gradient tensor is the inverse of the *material* deformation gradient tensor:

\[
\frac{\partial x_i}{\partial X_k} \frac{\partial X_k}{\partial x_j} = \delta_{ij} \quad \Rightarrow \quad F \cdot F^{-1} = F^{-1} \cdot F = 1
\]

- If \( F \) is not dependent on the space coordinates, \( F(X,t) \equiv F(t) \) the deformation is said to be **homogeneous**.
  - Every part of the solid body deforms as the whole does.
  - The associated motion is called **affine**.

- If there is **no motion**, \( x = X \) and \( F = \frac{\partial x}{\partial X} = F^{-1} = 1 \).
**Example**

Compute the deformation gradient and inverse deformation gradient tensors for a motion equation with Cartesian components given by,

\[
\begin{bmatrix}
X + Y^2 t \\
Y (1 + t) \\
Z e^t
\end{bmatrix}
\]

Using the results obtained, check that \( F \cdot F^{-1} = \mathbf{1} \).
Example - Solution

The Cartesian components of the deformation gradient tensor are,

\[
\begin{bmatrix}
F(X,t) = x \otimes \nabla \equiv x \nabla
\end{bmatrix}^T = \begin{bmatrix}
X + Y^2t \\
Y(1+t) \\
Ze^t
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}
\end{bmatrix} \begin{bmatrix}
1 & 2Yt & 0 \\
0 & 1+t & 0 \\
0 & 0 & e'
\end{bmatrix}
\]

The Cartesian components of the inverse motion equation will be given by,

\[
\begin{bmatrix}
X = x - \frac{y^2t}{(1+t)^2} \\
Y = \frac{y}{1+t} \\
Z = ze^{-t}
\end{bmatrix}
\begin{bmatrix}
F(X(x,t),t) \\
\mathbf{f}(x,t)
\end{bmatrix} = \begin{bmatrix}
\mathbf{f}(x,t)
\end{bmatrix} = \begin{bmatrix}
1 & \frac{2yt}{(1+t)} & 0 \\
0 & 1+t & 0 \\
0 & 0 & e'
\end{bmatrix}
\]
Example - Solution

The Cartesian components of the inverse deformation gradient tensor are,

\[
\begin{bmatrix}
F^{-1}(x,t) = \\
1 & -\frac{2yt}{(1+t)^2} & 0 \\
0 & \frac{1}{1+t} & 0 \\
0 & 0 & e^{-t}
\end{bmatrix}
\]

And it is verified that \( F \cdot F^{-1} = 1 \):

\[
F \cdot F^{-1} = \begin{bmatrix}
1 & \frac{2yt}{(1+t)} & 0 \\
0 & 1+t & 0 \\
0 & 0 & e^t
\end{bmatrix} \cdot \begin{bmatrix}
1 & -\frac{2yt}{(1+t)^2} & 0 \\
0 & \frac{1}{1+t} & 0 \\
0 & 0 & e^{-t}
\end{bmatrix} = \begin{bmatrix}
1 & -\frac{2yt}{(1+t)^2} + \frac{2yt}{(1+t)^2} & 0 \\
0 & \frac{1+t}{1+t} & 0 \\
0 & 0 & e^t e^{-t}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = [1]
\]
2.3 Displacements

Ch.2. Deformation and Strain
Displacements

- **Displacement**: relative position of a particle, in its current (deformed) configuration at time \( t \), with respect to its position in the initial (undeformed) configuration.

- **Displacement field**: displacement of all the particles in the continuous medium.

  - Material description (Lagrangian form):
    \[
    \begin{align*}
    U(X,t) &= x(X,t) - X \\
    U_i(X,t) &= x_i(X,t) - X_i & i \in \{1, 2, 3\}
    \end{align*}
    \]

  - Spatial description (Eulerian form):
    \[
    \begin{align*}
    u(x,t) &= x - X(x,t) \\
    u_i(x,t) &= x_i - X_i(x,t) & i \in \{1, 2, 3\}
    \end{align*}
    \]
Displacement Gradient Tensor

\[
\begin{align*}
\{U(X,t) = x(X,t) - X \\
U_i(X,t) = x_i(X,t) - X_i \quad i \in \{1,2,3\}
\end{align*}
\]

\[
\begin{aligned}
J_{ij} &= \frac{\partial U_i}{\partial X_j} = F_{ij} - \delta_{ij} \quad i,j \in \{1,2,3\} \\
J(X,t) &= U(X,t) \otimes \nabla = F - 1
\end{aligned}
\]

\[
\begin{align*}
u(x,t) = x - X(x,t) \\
u_i(x,t) = x_i - X_i(x,t) \quad i \in \{1,2,3\}
\end{align*}
\]

\[
\begin{aligned}
j_{ij} &= \frac{\partial u_i}{\partial x_j} = \delta_{ij} - F_{ij}^{-1} \quad i,j \in \{1,2,3\} \\
j(x,t) &= u(x,t) \otimes \nabla = 1 - F^{-1}
\end{aligned}
\]

**REMARK**  
If motion is a **pure shifting**: \( x(X,t) = X + U(t) \)  \( \Rightarrow \)  \( F = \frac{\partial x}{\partial X} = 1 = F^{-1} \) and  \( j = J = 0 \)
2.4 Strain Tensors

Ch. 2. Deformation and Strain
Strain Tensors

- **F** characterizes changes of relative placements during motion but is not a suitable measure of deformation for engineering purposes:
  - It is not null when no changes of distances and angles take place, e.g., in rigid-body motions.

- **Strain** is a normalized measure of deformation which characterizes the changes of distances and angles between particles:
  - It reduces to zero when there is no change of distances and angles between particles.
Consider

\[ dS = \sqrt{dX \cdot dX} \]
\[ ds = \sqrt{dx \cdot dx} \]

where \( dS \) is the length of segment \( dX \):

\[ dS = \sqrt{dX \cdot dX} \]

and \( ds \) is the length of segment \( dx \):

\[ ds = \sqrt{dx \cdot dx} \]
Strain Tensors

\[
\begin{align*}
\begin{aligned}
dX &= F^{-1} \cdot dx \\
x_i &= F_{ij}^{-1} \cdot dx_j
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
dS &= \sqrt{dX \cdot dX} \\
ds &= \sqrt{dx \cdot dx}
\end{align*}
\]

- One can write:

\[
\begin{align*}
\begin{aligned}
(ds)^2 &= dx \cdot dx = [dx]^T \cdot [dx] = [F \cdot dX]^T \cdot [F \cdot dX] = dX \cdot F^T \cdot F \cdot dX \\
(ds)^2 &= dx_k \cdot dx_k = F_{ki} \cdot dX_i \cdot F_{kj} \cdot dX_j = dX_i F_{ki} F_{kj} dX_j = dX_i F_{ki}^T F_{kj} dX_j
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
(dS)^2 &= dX \cdot dX = [dX]^T \cdot [dX] = [F^{-1} \cdot dx]^T \cdot [F^{-1} \cdot dx] = dx \cdot F^{-T} \cdot F^{-1} \cdot dx \\
(dS)^2 &= dX_k \cdot dX_k = F_{ki}^{-1} \cdot dx_i \cdot F_{kj}^{-1} \cdot dx_j = dX_i F_{ki}^{-1} F_{kj}^{-1} dx_j = dX_i F_{ik}^{-T} F_{kj}^{-1} dx_j
\end{aligned}
\end{align*}
\]

**REMARK**

The convention \[\begin{bmatrix} (\bullet)^{-1} \end{bmatrix}^T_{\text{not}} = (\bullet)^{-T}\] is used.
Green-Lagrange Strain Tensor

\[(ds)^2 = dX \cdot F^T \cdot F \cdot dX \quad (dS)^2 = dX \cdot dX\]

- Subtracting:
  \[(ds)^2 - (dS)^2 = dX \cdot F^T \cdot F \cdot dX - dX \cdot dX = dX \cdot F^T \cdot F \cdot dX - dX \cdot 1 \cdot dX = dX \cdot \left( F^T \cdot F - 1 \right) \cdot dX \]
  \[= 2 \cdot dX \cdot E \cdot dX \]

- The **Green-Lagrange** or **Material Strain Tensor** is defined:

\[
\begin{align*}
  E(X,t) &= \frac{1}{2} (F^T \cdot F - 1) \\
  E_{ij}(X,t) &= \frac{1}{2} (F_{ki} F_{kj} - \delta_{ij}) \quad i, j \in \{1, 2, 3\}
\end{align*}
\]

- \(E\) is symmetrical:

\[
\begin{align*}
  E^T &= \frac{1}{2} (F^T \cdot F - 1)^T = \frac{1}{2} \left( F^T \cdot \left( F^T \right)^T - 1 \right)^T = \frac{1}{2} \left( F^T \cdot F - 1 \right) = E \\
  E_{ij} &= E_{ji} \quad i, j \in \{1, 2, 3\}
\end{align*}
\]
Euler-Almansi Strain Tensor

\[(ds)^2 = dx \cdot dx \quad (dS)^2 = dx \cdot F^{-T} \cdot F^{-1} \cdot dx\]

- Subtracting:

\[(ds)^2 - (dS)^2 = dx \cdot dx - dx \cdot F^{-T} \cdot F^{-1} \cdot dx = dx \cdot 1 \cdot dx - dx \cdot F^{-T} \cdot F^{-1} \cdot dx = dx \cdot \left(1 - F^{-T} \cdot F^{-1}\right) \cdot dx = 2 dx \cdot e \cdot dx \]

\[
\text{def} = 2e
\]

- The **Euler-Almansi** or **Spatial Strain Tensor** is defined:

\[
\begin{align*}
\mathbf{e}(x, t) &= \frac{1}{2} \left(1 - F^{-T} \cdot F^{-1}\right) \\
\mathbf{e}_{ij}(x, t) &= \frac{1}{2} \left(\delta_{ij} - F^{-1}_{ki} F^{-1}_{kj}\right) \quad i, j \in \{1, 2, 3\}
\end{align*}
\]

- \(e\) is symmetrical:

\[
\begin{align*}
\mathbf{e}^T &= \frac{1}{2} \left(1 - F^{-T} \cdot F^{-1}\right)^T = \frac{1}{2} \left(1 - (F^{-1})^T \cdot (F^{-T})^T\right) = \frac{1}{2} \left(1 - F^{-T} \cdot F^{-1}\right) = e \\
\mathbf{e}_{ij} &= \mathbf{e}_{ji} \quad i, j \in \{1, 2, 3\}
\end{align*}
\]
The **Green-Lagrange** and the **Euler-Almansi Strain Tensors** are different tensors.

- They are **not** the material and spatial descriptions of a same strain tensor.
- They are affected by different vectors \(dX\) and \(dx\) when measuring distances:

\[
(ds)^2 - (dS)^2 = 2dX \cdot E \cdot dX = 2dx \cdot e \cdot dx
\]

The **Green-Lagrange Strain Tensor** is inherently obtained in material description, \(E = E\left(X, t\right)\).

- By substitution of the inverse Equations of Motion, \(E = E\left(X(x, t), t\right) = E(x, t)\).

The **Euler-Almansi Strain Tensor** is inherently obtained in spatial description, \(e = e\left(X, t\right)\).

- By substitution of the Equations of Motion, \(e = e\left(x(X, t), t\right) = e(X, t)\).
Strain Tensors in terms of Displacements

- Substituting \( F^{-1} = \mathbf{1} - \mathbf{j} \) and \( F = \mathbf{J} + \mathbf{1} \) into

\[
E = \frac{1}{2} \left( F^T \cdot F - \mathbf{1} \right) \quad \text{and} \quad e = \frac{1}{2} \left( \mathbf{1} - F^{-T} \cdot F^{-1} \right):
\]

\[
E = \frac{1}{2} \left[ (\mathbf{1} + \mathbf{J}^T) \cdot (\mathbf{1} + \mathbf{J}) - \mathbf{1} \right] = \frac{1}{2} \left[ \mathbf{J} + \mathbf{J}^T + \mathbf{J}^T \cdot \mathbf{J} \right]
\]

\[
E_{ij} = \frac{1}{2} \left[ \frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i} \frac{\partial U_k}{\partial X_j} \right] \quad i, j \in \{1, 2, 3\}
\]

\[
e = \frac{1}{2} \left[ \mathbf{1} - (\mathbf{1} - \mathbf{j}^T) \cdot (\mathbf{1} - \mathbf{j}) \right] = \frac{1}{2} \left[ \mathbf{j} + \mathbf{j}^T - \mathbf{j}^T \cdot \mathbf{j} \right]
\]

\[
e_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right] \quad i, j \in \{1, 2, 3\}
\]
Example

For the movement in the previous example, obtain the strain tensors in the material and spatial description.

\[
\begin{bmatrix}
X + Y^2t \\
Y(1+t) \\
Ze^t
\end{bmatrix}
\]
The deformation gradient tensor and its inverse tensor have already been obtained:

\[
F = \begin{bmatrix}
1 & 2Yt & 0 \\
0 & 1 + t & 0 \\
0 & 0 & e' \\
\end{bmatrix}
\quad
F^{-1} = \begin{bmatrix}
1 & \frac{2Yt}{(1+t)^2} & 0 \\
0 & \frac{1}{1+t} & 0 \\
0 & 0 & e^{-t} \\
\end{bmatrix}
\]

The material strain tensor:

\[
E = \frac{1}{2} (F^T \cdot F - I) = \frac{1}{2} \left( \begin{bmatrix}
1 & 0 & 0 \\
2Yt & 1 + t & 0 \\
0 & 0 & e' \\
\end{bmatrix} \cdot \begin{bmatrix}
1 & 2Yt & 0 \\
0 & 1 + t & 0 \\
0 & 0 & e' \\
\end{bmatrix} - I \right) = \frac{1}{2} \begin{bmatrix}
1 - 1 & 2Yt & 0 \\
2Yt & (2Yt)^2 + (1+t)^2 - 1 & 0 \\
0 & 0 & e'e' - 1 \\
\end{bmatrix}
\]

\[
= \frac{1}{2} \begin{bmatrix}
0 & 2Yt & 0 \\
2Yt & (2Yt)^2 + (1+t)^2 - 1 & 0 \\
0 & 0 & e^{2t} - 1 \\
\end{bmatrix}
\]
The spatial strain tensor:

\[
e = \frac{1}{2} \left( \mathbf{1} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} \right) = \frac{1}{2} \left( \mathbf{1} - \begin{pmatrix} 1 & 0 & 0 \\ \frac{2yt}{(1+t)^2} & \frac{1}{1+t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{2yt}{(1+t)^2} & 0 \\ 0 & \frac{1}{1+t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} 1 & -\frac{2yt}{(1+t)^2} & 0 \\ \frac{2yt}{(1+t)^2} & 1 - \left( -\frac{2yt}{(1+t)^2} \right)^2 + \left( \frac{1}{1+t} \right)^2 & 0 \\ 0 & 0 & 1 - e^{-t} e^{-t} \end{pmatrix}
\]
In conclusion, the material strain tensor is:

\[
E(x, t) = \frac{1}{2} \begin{bmatrix}
0 & 2yt & 0 \\
2yt & (2yt)^2 + (1+t)^2 - 1 & 0 \\
0 & 0 & e^{2t} - 1
\end{bmatrix}
\]

And the spatial strain tensor is:

\[
e(x, t) = \frac{1}{2} \begin{bmatrix}
0 & -\frac{2yt}{(1+t)^2} & 0 \\
-\frac{2yt}{(1+t)^2} & 1 - \left(\frac{2yt}{(1+t)^2}\right)^2 - \left(\frac{1}{1+t}\right)^2 & 0 \\
0 & 0 & 1 - e^{-2t}
\end{bmatrix}
\]

Observe that \(E(x, t) \neq e(x, t)\) and \(E(X, t) \neq e(X, t)\).
2.5 Variations of Distances

Ch.2. Deformation and Strain
The stretch ratio or stretch is defined as:

\[
\text{stretch} \overset{\text{def}}{=} \lambda_T = \lambda_t = \frac{P'Q'}{PQ} = \frac{ds}{dS} \quad (0 < \lambda < \infty)
\]

**REMARK**

The sub-indexes \(\bullet_T\) and \(\bullet_t\) are often dropped. But one must bear in mind that stretch and unit elongation always have a particular direction associated to them.
The extension or unit elongation is defined as:

\[
\text{unit elongation} \quad \text{def} \quad \varepsilon_T = \varepsilon_t = \frac{\Delta PQ}{PQ} = \frac{ds-dS}{dS}
\]

**REMARK**

The sub-indexes \((\bullet)_T\) and \((\bullet)_t\) are often dropped. But one must bear in mind that stretch and unit elongation always have a particular direction associated to them.
The **stretch** and **unit elongation** for a same point and direction are related through:

\[ \varepsilon = \frac{ds - dS}{dS} = \left( \frac{ds}{dS} \right) - 1 = \lambda - 1 \implies (-1 < \varepsilon < \infty) \]

- If \( \lambda = 1 \) \( (\varepsilon = 0) \) \( \implies ds = dS \): \( P \) and \( Q \) may have moved in time but have kept the distance between them constant.

- If \( \lambda > 1 \) \( (\varepsilon > 0) \) \( \implies ds > dS \): the distance between them \( P \) and \( Q \) has increased with the deformation of the medium.

- If \( \lambda < 1 \) \( (\varepsilon < 0) \) \( \implies ds < dS \): the distance between them \( P \) and \( Q \) has decreased with the deformation of the medium.
Stretch and Unit Elongation in terms of the Strain Tensors

- Considering:

\[
\begin{align*}
(ds)^2 - (dS)^2 &= 2 dX \cdot E \cdot dX \\
(ds)^2 - (dS)^2 &= 2 dx \cdot e \cdot dx
\end{align*}
\]

- Then:

\[
\begin{align*}
(ds)^2 - (dS)^2 &= 2(ds)^2 T \cdot E \cdot T \\
(ds)^2 - (dS)^2 &= 2(ds)^2 t \cdot e \cdot t
\end{align*}
\]

\[
\lambda = \sqrt{1 + 2 T \cdot E \cdot T} \\
\varepsilon = \lambda - 1 = \sqrt{1 + 2 T \cdot E \cdot T} - 1
\]

\[
\lambda = \frac{1}{\sqrt{1 - 2 t \cdot e \cdot t}} \\
\varepsilon = \lambda - 1 = \frac{1}{\sqrt{1 - 2 t \cdot e \cdot t}} - 1
\]

**REMARK**

\(E(X,t)\) and \(e(x,t)\) contain information regarding the stretch and unit elongation for any direction in the differential neighborhood of a point.
2.6 Variation of Angles

Ch.2. Deformation and Strain
Variation of Angles

\[
\begin{align*}
    dX^{(1)} &= T^{(1)} \, dS^{(1)} \\
    dX^{(2)} &= T^{(2)} \, dS^{(2)}
\end{align*}
\]

The scalar product of the vectors $dX^{(1)}$ and $dX^{(2)}$:

\[
dX^{(1)} \cdot dX^{(2)} = \left| dx^{(1)} \right| \cdot \left| dx^{(2)} \right| \cos \theta = ds^{(1)} \, ds^{(2)} \cos \theta
\]
Variation of Angles

\[
\frac{dx^{(1)}}{ds^{(1)}} \cdot \frac{dx^{(2)}}{ds^{(2)}} = \cos \theta \cdot ds^{(1)} ds^{(2)}
\]

\[
dx^{(1)} = F \cdot dX^{(1)}
\]
\[
dx^{(2)} = F \cdot dX^{(2)}
\]

\[
dx^{(1)} \cdot dx^{(2)} = \left[ F \cdot dX^{(1)} \right]^T \cdot \left[ F \cdot dX^{(2)} \right] = dX^{(1)} \cdot \left( F^T \cdot F \right) \cdot dX^{(2)}
\]

\[
dX^{(1)} = T^{(1)} \cdot dS^{(1)}
\]
\[
dX^{(2)} = T^{(2)} \cdot dS^{(2)}
\]

\[
dx^{(1)} \cdot dx^{(2)} = \frac{dS^{(1)}}{\lambda^{(1)}} \cdot T^{(1)} \cdot (2E + 1) \cdot T^{(2)} \cdot \frac{dS^{(2)}}{\lambda^{(2)}} = ds^{(1)} ds^{(2)} \left( \frac{1}{\lambda^{(1)}} \right) \cdot \left( \frac{1}{\lambda^{(2)}} \right) \cdot T^{(1)} \cdot (2E + 1) \cdot T^{(2)} = ds^{(1)} ds^{(2)} \cos \theta
\]

\[
\cos \theta = \frac{T^{(1)} \cdot (1 + 2E) \cdot T^{(2)}}{\sqrt{1 + 2} \cdot T^{(1)} \cdot E \cdot T^{(1)} \cdot \sqrt{1 + 2} \cdot T^{(2)} \cdot E \cdot T^{(2)}}
\]

\[
\lambda = \sqrt{1 + 2} \cdot T \cdot E \cdot T
\]

\[
\left( \frac{T^{(1)} \cdot (1 + 2E) \cdot T^{(2)}}{\sqrt{1 + 2} \cdot T^{(1)} \cdot E \cdot T^{(1)} \cdot \sqrt{1 + 2} \cdot T^{(2)} \cdot E \cdot T^{(2)}} \right)
\]
The scalar product of the vectors $dX^{(1)}$ and $dX^{(2)}$:

$$dX^{(i)} \cdot dX^{(2)} = |dX^{(i)}| \cdot |dX^{(2)}| \cos \Theta = dS^{(i)} \cdot dS^{(2)} \cos \Theta$$
Variation of Angles

\[ \frac{dX^{(i)}}{dx^{(i)}} \cdot \frac{dX^{(2)}}{dx^{(2)}} = dS^{(1)} \cdot dS^{(2)} \cos \Theta \]

\[ \begin{cases} 
  dx^{(i)} = F^{-1} \cdot dx^{(i)} \\
  dx^{(2)} = F^{-1} \cdot dx^{(2)} 
\end{cases} \]

\[ dX^{(i)} \cdot dX^{(2)} = \begin{bmatrix} F^{-1} \cdot dx^{(i)} \end{bmatrix}^T \begin{bmatrix} F^{-1} \cdot dx^{(2)} \end{bmatrix} = dx^{(i)} \cdot \frac{(F^{-T} \cdot F^{-1}) \cdot dx^{(2)}}{1-2e} \]

\[ \begin{cases} 
  dx^{(1)} = t^{(1)} ds^{(1)} \\
  dx^{(2)} = t^{(2)} ds^{(2)} 
\end{cases} \]

\[ dX^{(i)} \cdot dX^{(2)} = \frac{ds^{(i)}}{\lambda^{(i)} ds^{(i)}} t^{(i)} \cdot (1 - 2e) \cdot t^{(2)} \frac{ds^{(2)}}{\lambda^{(2)} ds^{(2)}} = dS^{(1)} \cdot dS^{(2)} \left( \lambda^{(i)} \lambda^{(2)} t^{(i)} \cdot (1 - 2e) \cdot t^{(2)} \right) = dS^{(1)} \cdot dS^{(2)} \cos \Theta \]

\[ \lambda = \frac{1}{\sqrt{1 - 2t^{(i)} \cdot e \cdot t^{(i)}}} \]

\[ \left( \frac{t^{(i)} \cdot (1 - 2e) \cdot t^{(2)}}{\sqrt{1 - 2t^{(1)} \cdot e \cdot t^{(1)}} \sqrt{1 - 2t^{(2)} \cdot e \cdot t^{(2)}}} \right) \]

**REMARK**

E(x,t) and e(x,t) contain information regarding the variation in angles between segments in the differential neighbourhood of a point.
Example

Let us consider the motion of a continuum body such that the spatial description of the Cartesian components of the spatial Almansi strain tensor is given by,

\[
[e(x, t)] = \begin{bmatrix}
0 & 0 & -te^{tz} \\
0 & 0 & 0 \\
-te^{tz} & 0 & t\left(2e^{tz} - e^t\right)
\end{bmatrix}
\]

Compute at time \(t=0\) (the reference time), the length of the curve that at time \(t=2\) is a straight line going from point \(a\) \((0,0,0)\) to point \(b\) \((1,1,1)\).

The length of the curve at time \(t=0\) can be expressed as,

\[
L = \int_{A}^{B} \frac{dS}{\lambda} = \int_{a}^{b} \frac{1}{\lambda}(x, t)\, ds
\]
Example - Solution

The inverse of the stretch, at the points belonging to the straight line going from \(a(0,0,0)\) to \(b(1,1,1)\) along the unit vector in the direction of the straight line, is given by,

\[
\lambda(x,t) = \frac{1}{\sqrt{1 - 2t \cdot e(x,t) \cdot t}} \quad \rightarrow \quad \lambda^{-1}(x,t) = \sqrt{1 - 2t \cdot e(x,t) \cdot t}
\]

Where the unit vector is given by,

\[
[t] = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T
\]

Substituting the unit vector and spatial Almansi strain tensor into the expression of the inverse of the stretching yields,

\[
\lambda^{-1}(x,t) = \sqrt{1 + \frac{2}{3} te^t}
\]
Example - Solution

The inverse of the stretch, which is uniform and therefore does not depend on the spatial coordinates, at time $t=2$ reads,

$$\lambda^{-1}(x, 2) = \sqrt{1 + \frac{4}{3}e^2}$$

Substituting the inverse of the stretch into the integral expression provides the length at time $t=0$,

$$L = \int_{\Gamma} dS = \int_{a}^{b} \lambda^{-1}(x, 2) ds = \int_{a}^{b} \sqrt{1 + \frac{4}{3}e^2} ds = \sqrt{1 + \frac{4}{3}e^2} \int_{(0,0,0)}^{(1,1,1)} ds = \sqrt{3 + 4e^2}$$
2.7 Physical Interpretation of $E$ and $e$
Physical Interpretation of $E$

- Consider the components of the material strain tensor, $E$:

$$E = \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{XY} & E_{YY} & E_{YZ} \\ E_{XZ} & E_{YZ} & E_{ZZ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}$$

- For a segment parallel to the $X$-axis, the stretch is:

$$\lambda = \sqrt{1 + 2T \cdot E \cdot T}$$

$$T^{(1)} \cdot E \cdot T^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = E_{11}$$

Stretching of the material in the $X$-direction:

$$\lambda_1 = \sqrt{1 + 2E_{11}}$$
Physical Interpretation of $E$

- Similarly, the stretching of the material in the $Y$-direction and the $Z$-direction:

\[ \lambda_1 = \sqrt{1 + 2E_{11}} \quad \Rightarrow \quad \varepsilon_X = \lambda_X - 1 = \sqrt{1 + 2E_{XX}} - 1 \]
\[ \lambda_2 = \sqrt{1 + 2E_{22}} \quad \Rightarrow \quad \varepsilon_Y = \lambda_Y - 1 = \sqrt{1 + 2E_{YY}} - 1 \]
\[ \lambda_3 = \sqrt{1 + 2E_{33}} \quad \Rightarrow \quad \varepsilon_Z = \lambda_Z - 1 = \sqrt{1 + 2E_{ZZ}} - 1 \]

- The **longitudinal strains** contain information on the stretch and unit elongation of the segments initially oriented in the $X$, $Y$ and $Z$-directions (in the material configuration).

\[
E = \begin{bmatrix}
E_{XX} & E_{XY} & E_{XZ} \\
E_{XY} & E_{YY} & E_{YZ} \\
E_{XZ} & E_{YZ} & E_{ZZ}
\end{bmatrix}
\]

- If $E_{XX} = 0$ \quad $\Rightarrow \quad \varepsilon_X = 0$ \quad $\Rightarrow$ No elongation in the $X$-direction
- If $E_{YY} = 0$ \quad $\Rightarrow \quad \varepsilon_Y = 0$ \quad $\Rightarrow$ No elongation in the $Y$-direction
- If $E_{ZZ} = 0$ \quad $\Rightarrow \quad \varepsilon_Z = 0$ \quad $\Rightarrow$ No elongation in the $Z$-direction
Physical Interpretation of $E$

Consider the angle between a segment parallel to the $X$-axis and a segment parallel to the $Y$-axis, the angle is:

$$\theta = \theta_{xy} = \arccos \frac{2E_{xy}}{\sqrt{1+2E_{xx}} \sqrt{1+2E_{yy}}} = \frac{\pi}{2} - \arcsin \frac{2E_{xy}}{\sqrt{1+2E_{xx}} \sqrt{1+2E_{yy}}}$$

$$\cos \theta = \frac{T^{(i)} \cdot (1+2E) \cdot T^{(2)}}{\sqrt{1+2} \ T^{(i)} \cdot E \cdot T^{(i)} \cdot \sqrt{1+2} \ T^{(2)} \cdot E \cdot T^{(2)}}$$

$$T^{(i)} = \begin{cases} 1 \\ 0 \\ 0 \end{cases}, \quad T^{(2)} = \begin{cases} 0 \\ 0 \\ 1 \end{cases}$$

$$T^{(i)} \cdot T^{(2)} = 0$$

$$T^{(i)} \cdot E \cdot T^{(i)} = E_{11}$$

$$T^{(i)} \cdot E \cdot T^{(2)} = E_{12}$$

$$T^{(2)} \cdot E \cdot T^{(2)} = E_{22}$$

$$cos \theta = \frac{2E_{12}}{\sqrt{1+2E_{11}} \sqrt{1+2E_{22}}}$$
Physical Interpretation of $E$

\[ \theta \equiv \theta_{xy} = \frac{\pi}{2} - \arcsin \frac{2E_{xy}}{\sqrt{1+2E_{xx}} \sqrt{1+2E_{yy}}} \]

- The increment of the final angle w.r.t. its initial value:

\[ \Delta \Theta_{xy} = \theta_{xy} - \Theta_{xy} = -\arcsin \frac{2E_{xy}}{\sqrt{1+2E_{xx}} \sqrt{1+2E_{yy}}} \]
Physical Interpretation of \( E \)

- Similarly, the increment of the final angle w.r.t. its initial value for couples of segments oriented in the direction of the coordinate axes:

\[
\begin{align*}
\Delta \Theta_{xy} &= -\arcsin \frac{2E_{xy}}{\sqrt{1+2E_{xx}} \sqrt{1+2E_{yy}}} \\
\Delta \Theta_{xz} &= -\arcsin \frac{2E_{xz}}{\sqrt{1+2E_{xx}} \sqrt{1+2E_{zz}}} \\
\Delta \Theta_{yz} &= -\arcsin \frac{2E_{yz}}{\sqrt{1+2E_{yy}} \sqrt{1+2E_{zz}}}
\end{align*}
\]

- The **angular strains** contain information on the variation of the angles between segments initially oriented in the \( X, Y \) and \( Z \)-directions (in the material configuration).

\[
E = \begin{bmatrix}
E_{xx} & E_{xy} & E_{xz} \\
E_{xy} & E_{yy} & E_{yz} \\
E_{xz} & E_{yz} & E_{zz}
\end{bmatrix}
\]

- If \( E_{xy} = 0 \) \( \Rightarrow \) No angle variation between the \( X- \) and \( Y \)-directions
- If \( E_{xz} = 0 \) \( \Rightarrow \) No angle variation between the \( X- \) and \( Z \)-directions
- If \( E_{yz} = 0 \) \( \Rightarrow \) No angle variation between the \( Y- \) and \( Z \)-directions
Physical Interpretation of $E$

In short,

- $\lambda_1 \, dX = \sqrt{1 + 2E_{xx}} \, dX$
- $\lambda_2 \, dY = \sqrt{1 + 2E_{yy}} \, dY$
- $\lambda_3 \, dZ = \sqrt{1 + 2E_{zz}} \, dZ$

\[
\Delta \Theta_{xy} = -\arcsin \frac{2E_{xy}}{\sqrt{1 + 2E_{xx}} \sqrt{1 + 2E_{yy}}}
\]

\[
\Delta \Theta_{xz} = -\arcsin \frac{2E_{xz}}{\sqrt{1 + 2E_{xx}} \sqrt{1 + 2E_{zz}}}
\]

\[
\Delta \Theta_{yz} = -\arcsin \frac{2E_{yz}}{\sqrt{1 + 2E_{yy}} \sqrt{1 + 2E_{zz}}}
\]
Physical Interpretation of $e$

- Consider the components of the spatial strain tensor, $e$:

$$e = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix}$$

- For a segment parallel to the $x$-axis, the stretch is:

$$\lambda = \frac{1}{\sqrt{1 - 2 t \cdot e \cdot t}}$$

For $t^{(1)} = \hat{e}_1$

$$t^{(1)} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad dx \equiv \begin{bmatrix} ds \\ 0 \\ 0 \end{bmatrix}$$

$$t \cdot e \cdot t = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_{11}$$

$$\lambda_1 = \frac{1}{\sqrt{1 - 2e_{11}}}$$

Stretching of the material in the $x$-direction
Similarly, the stretching of the material in the \( y \)-direction and the \( z \)-direction:

\[
\begin{align*}
\lambda_1 &= \frac{1}{\sqrt{1-2e_{11}}} \quad \Rightarrow \quad \varepsilon_x = \lambda_x - 1 = \frac{1}{\sqrt{1-2e_{xx}}} - 1 \\
\lambda_2 &= \frac{1}{\sqrt{1-2e_{22}}} \quad \Rightarrow \quad \varepsilon_y = \lambda_y - 1 = \frac{1}{\sqrt{1-2e_{yy}}} - 1 \\
\lambda_3 &= \frac{1}{\sqrt{1-2e_{33}}} \quad \Rightarrow \quad \varepsilon_z = \lambda_z - 1 = \frac{1}{\sqrt{1-2e_{zz}}} - 1
\end{align*}
\]

The **longitudinal strains** contain information on the stretch and unit elongation of the segments oriented in the \( x \), \( y \) and \( z \)-directions (in the deformed or actual configuration).
Consider the angle between a segment parallel to the $x$-axis and a segment parallel to the $y$-axis, the angle is:

\[
\cos \Theta = \frac{t^{(1)} \cdot (1 - 2e) \cdot t^{(2)}}{\sqrt{1 - 2} \cdot e \cdot t^{(1)} \cdot \sqrt{1 - 2} \cdot e \cdot t^{(2)}}
\]

\[
\begin{align*}
t^{(1)} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
t^{(2)} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
t^{(1)} \cdot t^{(2)} &= 0 \\
t^{(1)} \cdot e \cdot t^{(1)} &= e_{11} \\
t^{(1)} \cdot e \cdot t^{(2)} &= e_{12} \\
t^{(2)} \cdot e \cdot t^{(2)} &= e_{22}
\end{align*}
\]

\[
\cos \Theta = \frac{-2 e_{12}}{\sqrt{1 - 2 e_{11}} \cdot \sqrt{1 - 2 e_{22}}}
\]

\[
\Theta \equiv \Theta_{xy} = \frac{\pi}{2} + \arcsin \left( \frac{2 e_{xy}}{\sqrt{1 - 2 e_{xx}} \cdot \sqrt{1 - 2 e_{yy}}} \right)
\]
The increment of the angle in the reference configuration w.r.t. its value in the deformed one:

\[
\Delta \theta_{xy} = \theta_{xy} - \Theta_{xy} = -\arcsin \frac{2e_{xy}}{\sqrt{1-2e_{xx}} \sqrt{1-2e_{yy}}}.
\]
Similarly, the increment of the angle in the reference configuration w.r.t. its value in the deformed one for couples of segments oriented in the direction of the coordinate axes:

\[
\Delta \theta_{xy} = \frac{\pi}{2} - \Theta_{xy} = -\arcsin \frac{2e_{xy}}{\sqrt{1-2e_{xx}} \sqrt{1-2e_{yy}}}
\]

\[
\Delta \theta_{xz} = \frac{\pi}{2} - \Theta_{xz} = -\arcsin \frac{2e_{xz}}{\sqrt{1-2e_{xx}} \sqrt{1-2e_{zz}}}
\]

\[
\Delta \theta_{yz} = \frac{\pi}{2} - \Theta_{yz} = -\arcsin \frac{2e_{yz}}{\sqrt{1-2e_{yy}} \sqrt{1-2e_{zz}}}
\]

The angular strains contain information on the variation of the angles between segments oriented in the x, y and z-directions (in the deformed or actual configuration).

\[
e \equiv \begin{bmatrix}
e_{xx} & e_{xy} & e_{xz} \\
e_{xy} & e_{yy} & e_{yz} \\
e_{xz} & e_{yz} & e_{zz}
\end{bmatrix}
\]
Physical Interpretation of $e$

- In short,

\[
\begin{align*}
\frac{dx}{\lambda_1} &= \sqrt{1 - 2e_{xx}} \ dx \\
\frac{dy}{\lambda_2} &= \sqrt{1 - 2e_{yy}} \ dy \\
\frac{dz}{\lambda_3} &= \sqrt{1 - 2e_{zz}} \ dz
\end{align*}
\]

\[
\Delta \theta_{xy} = \frac{\pi}{2} - \Theta_{xy} = -\arcsin \frac{2e_{xy}}{\sqrt{1 - 2e_{xx}} \sqrt{1 - 2e_{yy}}}
\]

\[
\Delta \theta_{xz} = \frac{\pi}{2} - \Theta_{xz} = -\arcsin \frac{2e_{xz}}{\sqrt{1 - 2e_{xx}} \sqrt{1 - 2e_{zz}}}
\]

\[
\Delta \theta_{yz} = \frac{\pi}{2} - \Theta_{yz} = -\arcsin \frac{2e_{yz}}{\sqrt{1 - 2e_{yy}} \sqrt{1 - 2e_{zz}}}
\]
2.8 Polar Decomposition

Ch.2. Deformation and Strain
Polar Decomposition

Polar Decomposition Theorem:

“For any non-singular 2\textsuperscript{nd} order tensor $F$ there exist two unique positive-definite symmetrical 2\textsuperscript{nd} order tensors $U$ and $V$, and a unique orthogonal 2\textsuperscript{nd} order tensor $Q$ such that: ”

\begin{align*}
U &= \sqrt{F^T \cdot F} \\
V &= \sqrt{F \cdot F^T} \\
Q &= F \cdot U^{-1} = V^{-1} \cdot F
\end{align*}

The decomposition is unique.

- $Q$: Rotation tensor
- $U$: Right or material stretch tensor
- $V$: Left or spatial stretch tensor

**REMARK**

An orthogonal 2\textsuperscript{nd} order tensor verifies:

$$Q^T \cdot Q = Q \cdot Q^T = 1$$
Properties of an orthogonal tensor

- An **orthogonal tensor** $Q$ when multiplied (dot product) times a vector rotates it (without changing its length): $y = Q \cdot x$
  - $y$ has the same norm as $x$:
    \[
    \|y\|^2 = y \cdot y = [y]^T [y] = [Q \cdot x]^T [Q \cdot x] = x \cdot Q^T \cdot Q \cdot x = \|x\|^2
    \]
  - when $Q$ is applied on two vectors $x^{(1)}$ and $x^{(2)}$, with the same origin, the original angle they form is maintained:
    \[
    y^{(1)} = Q \cdot x^{(1)} \quad y^{(2)} = Q \cdot x^{(2)} \\
    \frac{y^{(1)} \cdot y^{(2)}}{\|y^{(1)}\| \|y^{(2)}\|} = \frac{x^{(1)} \cdot Q^T \cdot Q \cdot x^{(2)}}{\|y^{(1)}\| \|y^{(2)}\|} = \frac{x^{(1)} \cdot x^{(2)}}{\|x^{(1)}\| \|x^{(2)}\|} = \cos \alpha
    \]
- Consequently, the rotation $y = Q \cdot x$ maintains angles and distances.
Polar Decomposition of $\mathbf{F}$

Consider the deformation gradient tensor, $\mathbf{F}$:

\[
\mathbf{F} = \mathbf{Q} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{Q}
\]

\[
d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = (\mathbf{V} \cdot \mathbf{Q}) \cdot d\mathbf{X} = \mathbf{V} \cdot (\mathbf{Q} \cdot d\mathbf{X})
\]

\[
\mathbf{F}(\bullet) \equiv \text{stretching } \circ \text{rotation}(\bullet)
\]

\[
d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = (\mathbf{Q} \cdot \mathbf{U}) \cdot d\mathbf{X} = \mathbf{Q} \cdot (\mathbf{U} \cdot d\mathbf{X})
\]

\[
\mathbf{F}(\bullet) \equiv \text{rotation } \circ \text{stretching}(\bullet)
\]

**REMARK**

For a rigid body motion:

\[
\mathbf{U} = \mathbf{V} = \mathbf{1} \quad \text{and} \quad \mathbf{Q} = \mathbf{F}
\]
2.9 Volume Variation

Ch. 2. Deformation and Strain
Consider the variation of a differential volume associated to a particle P:

\[
dV_0 = \left( d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} \right) \cdot d\mathbf{X}^{(3)} = \begin{vmatrix} d\mathbf{x}_1^{(1)} & d\mathbf{x}_2^{(1)} & d\mathbf{x}_3^{(1)} \\ d\mathbf{x}_1^{(2)} & d\mathbf{x}_2^{(2)} & d\mathbf{x}_3^{(2)} \\ d\mathbf{x}_1^{(3)} & d\mathbf{x}_2^{(3)} & d\mathbf{x}_3^{(3)} \end{vmatrix} = |\mathbf{M}| \]

\[
dV_t = \left( d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} \right) \cdot d\mathbf{x}^{(3)} = \begin{vmatrix} d\mathbf{x}_1^{(1)} & d\mathbf{x}_2^{(1)} & d\mathbf{x}_3^{(1)} \\ d\mathbf{x}_1^{(2)} & d\mathbf{x}_2^{(2)} & d\mathbf{x}_3^{(2)} \\ d\mathbf{x}_1^{(3)} & d\mathbf{x}_2^{(3)} & d\mathbf{x}_3^{(3)} \end{vmatrix} = |\mathbf{m}| \]

\[
M_{ij} = dX_j^{(i)} \quad m_{ij} = dx_j^{(i)}
\]
Differential Volume Ratio

Consider now:

\[\begin{align*}
    dx_i^{(i)} &= F \cdot dX_i^{(i)} \quad &i \in \{1, 2, 3\} &\rightarrow \text{Fundamental eq. of deformation} \\
    dx_j^{(i)} &= F_{jk} \cdot dX_k^{(i)} \\
\end{align*}\]

And, defining \( J(X,t) \) as the jacobian of the deformation,

\( J(X,t) = \det F(X,t) > 0 \)

\[dV_t = J \cdot dV_0\]
2.10 Area Variation

Ch.2. Deformation and Strain
Consider the variation of a differential area associated to a particle $P$:

\[
dA := dAN \rightarrow \text{material vector "differential of area"} \rightarrow |dA| = dA
\]

\[
da := dan \rightarrow \text{spatial vector "differential of area"} \rightarrow |da| = dA
\]

\[
dV_0 = dH \, dA = \frac{\mathbf{dX}^{(3)} \cdot \mathbf{N}}{dH} \, dA = \mathbf{dX}^{(3)} \cdot \mathbf{N} \, dA = dA \cdot d\mathbf{X}^{(3)}
\]

\[
dV_t = dh \, da = \frac{\mathbf{dX}^{(3)} \cdot \mathbf{n}}{dh} \, da = \mathbf{dX}^{(3)} \cdot \mathbf{n} \, da = da \cdot d\mathbf{X}^{(3)}
\]
Consider now:

\[ dV_t = da \cdot dx^{(3)} \]

\[ \begin{aligned}
    dx^{(3)} &= F \cdot dX^{(3)} \\
    dV_t &= |F| \cdot dV_0 \\
    dV_0 &= dA \cdot dX^{(3)}
\end{aligned} \]

\[ \begin{aligned}
    dV_t &= |F| \cdot dA \cdot dX^{(3)} \\
    dV_0 &= |F| \cdot dV_0
\end{aligned} \]

\[ \begin{aligned}
    da &= |F| \cdot dA \cdot F^{-1} \\
    da \cdot n &= |F| \cdot N \cdot F^{-1} \cdot dA \\
    da &= |F| \cdot N \cdot F^{-1} \cdot dA \\
    dA &= N \cdot dA \\
    da &= n \cdot da
\end{aligned} \]
2.11 Volumetric Strain

Ch.2. Deformation and Strain
Volumetric Strain:

\[
e(\mathbf{X}, t) \overset{\text{def}}{=} \frac{dV(\mathbf{X}, t) - dV(\mathbf{X}, t_0)}{dV(\mathbf{X}, t)} = \frac{dV_t - dV_0}{dV_0}
\]

\[
dV_t = |\mathbf{F}|dV_0
\]

\[
e = \frac{|\mathbf{F}|dV_0 - dV_0}{dV_0} = |\mathbf{F}| - 1
\]

REMARK
The incompressibility condition (null volumetric strain) takes the form

\[
e = J - 1 = 0 \implies J = |\mathbf{F}| = 1
\]
2.12 Infinitesimal Strain

Ch.2. Deformation and Strain
The **infinitesimal strain theory** (also called **small strain theory**) is based on the simplifying hypotheses:

- Displacements are very small w.r.t. the typical dimensions in the continuum medium,

- As a consequence, $\|u\| \ll \text{(size of } \Omega_0)$ and the reference and deformed configurations are considered to be practically the same, as are the material and spatial coordinates:

  \[ \begin{align*}
  \Omega &\cong \Omega_0 \quad \text{and} \\
  x &= X + u \cong X, \\
  x_i &= X_i + u_i \cong X_i
  \end{align*} \]

- Displacement gradients are infinitesimal,

  \[ \left| \frac{\partial u_i}{\partial x_j} \right| \ll 1, \quad \forall i, j \in \{1,2,3\} \]
Infinitesimal Strain Theory

- The material and spatial coordinates coincide, \( x = X + u \approx 0 \)
  - Even though it is considered that \( u \) cannot be neglected when calculating other properties such as the infinitesimal strain tensor \( \varepsilon \).

- There is no difference between the material and spatial differential operators:
  \[
  \nabla^{\text{symb}} = \frac{\partial}{\partial X_i} \mathbf{\hat{e}}_i = \frac{\partial}{\partial x_i} \mathbf{\hat{e}}_i = \nabla
  \]
  \[
  J(X,t) = U(X,t) \otimes \nabla = u(x,t) \otimes \nabla = j(x,t)
  \]

- The local an material time derivatives coincide
  \[
  \Gamma(X,t) \approx \Gamma(x,t) = \gamma(x,t) = \gamma(X,t)
  \]
  \[
  \frac{d\gamma}{dt} = \frac{\partial \gamma(X,t)}{\partial t} = \frac{\partial \gamma(x,t)}{\partial t} = \dot{\gamma}
  \]
Strain Tensors

- **Green-Lagrange strain tensor**

\[
E = \frac{1}{2} (F^T F - I) = \frac{1}{2} (J + J^T + 2J^T J)
\]

\[
E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right)
\]

Therefore, the **infinitesimal strain tensor** is defined as:

\[
\varepsilon = \frac{1}{2} (J + J^T) = \frac{1}{2} (j + j^T) = \nabla \otimes \nabla \otimes u = \nabla^s u \]

\[
\varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \quad i, j \in \{1, 2, 3\}
\]

**Remark**

\[
\varepsilon \text{ is a symmetrical tensor and its components are infinitesimal: } |\varepsilon_{ij}| \ll 1, \quad \forall i, j \in \{1, 2, 3\}
\]

- **Euler-Almansi strain tensor**

\[
E \approx \frac{1}{2} (J + J^T) = \frac{1}{2} (j + j^T) = \varepsilon
\]

\[
E_{ij} \approx \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \varepsilon_{ij} \quad i, j \in \{1, 2, 3\}
\]
Stretch and Unit Elongation

- Stretch in terms of the strain tensors:
  \[ \lambda_T = \sqrt{1 + 2 \frac{T \cdot E \cdot T}{x}} \]

But in Infinitesimal Strain Theory, \( T \approx t \). So the linearized stretch and unit elongation through a direction given by the unit vector \( T \approx t \) are:

- \( \lambda = \frac{ds}{dS} \approx 1 + T \cdot \varepsilon \cdot t \approx 1 + T \cdot \varepsilon \cdot T \)
- \( \varepsilon = \frac{ds - dS}{dS} = \lambda - 1 = t \cdot \varepsilon \cdot t \)
Physical Interpretation of Infinitesimal Strains

- Consider the components of the infinitesimal strain tensor, $\varepsilon$:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \equiv \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

- For a segment parallel to the $x$-axis, the stretch and unit elongation are:

$$\lambda \equiv 1 + t \cdot \varepsilon \cdot t$$

$$t \cdot \varepsilon \cdot t = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \varepsilon_{11}$$

$$\lambda_1 = \lambda_x = 1 + \varepsilon_{11}$$

$$\varepsilon_1 = 1 - \lambda = \varepsilon_{11}$$

$$\varepsilon_x = 1 - \lambda = \varepsilon_{xx}$$

Stretch in the $x$-direction

Unit elongation in the $x$-direction
Physical Interpretation of Infinitesimal Strains

- Similarly, the stretching and unit elongation of the material in the $y$-direction and the $z$-direction:

$$\lambda_1 = 1 + \varepsilon_{11} \quad \Rightarrow \quad \varepsilon_x = \lambda_x - 1 = \varepsilon_{xx}$$
$$\lambda_2 = 1 + \varepsilon_{22} \quad \Rightarrow \quad \varepsilon_y = \lambda_y - 1 = \varepsilon_{yy}$$
$$\lambda_3 = 1 + \varepsilon_{33} \quad \Rightarrow \quad \varepsilon_z = \lambda_z - 1 = \varepsilon_{zz}$$

- The diagonal components of the infinitesimal strain tensor are the unit elongations of the material when in the $x$, $y$ and $z$-directions.

$$\varepsilon = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}
\end{bmatrix}$$
Physical Interpretation of Infinitesimal Strains

- Consider the angle between a segment parallel to the \( X \)-axis and a segment parallel to the \( Y \)-axis, the angle is \( \Theta_{xy} = \frac{\pi}{2} \).

- Applying:

\[
\theta \equiv \theta_{xy} = \frac{\pi}{2} - \arcsin \frac{2E_{xy}}{\sqrt{1+2E_{xx}} \sqrt{1+2E_{yy}}}
\]

\[
\left\{ \begin{align*}
E_{xx} &= \varepsilon_{xx} \\
E_{xy} &= \varepsilon_{xy} \\
E_{yy} &= \varepsilon_{yy}
\end{align*} \right.
\]

\[
\theta_{xy} = \frac{\pi}{2} - \arcsin \frac{2\varepsilon_{xy}}{\sqrt{1+2\varepsilon_{xx}} \sqrt{1+2\varepsilon_{yy}}} \approx \frac{\pi}{2} - \arcsin 2\varepsilon_{xy} = \frac{\pi}{2} - 2\varepsilon_{xy}
\]

**REMARK**

The Taylor linear series expansion of \( \arcsin x \) yields

\[
\arcsin(x) \approx \arcsin(0) + \frac{d}{dx} \arcsin(x) \bigg|_{x=0} (x) + \ldots = x + O(x^2)
\]
Physical Interpretation of Infinitesimal Strains

\[ \theta_{xy} \approx \frac{\pi}{2} - 2\varepsilon_{xy} \]

- The increment of the final angle w.r.t. its initial value:

\[ \Delta \theta_{xy} = \theta_{xy} - \frac{\pi}{2} \approx \frac{\pi}{2} - 2\varepsilon_{xy} - \frac{\pi}{2} = -2\varepsilon_{xy} \]

- Similarly, the increment of the final angle w.r.t. its initial value for couples of segments oriented in the direction of the coordinate axes:

\[ \varepsilon_{xy} = -\frac{1}{2} \Delta \theta_{xy} \quad ; \quad \varepsilon_{xz} = -\frac{1}{2} \Delta \theta_{xz} \quad ; \quad \varepsilon_{yz} = -\frac{1}{2} \Delta \theta_{yz} \]

- The non-diagonal components of the infinitesimal strain tensor are equal to the semi-decrements produced by the deformation of the angles between segments initially oriented in the x, y and z-directions.
Physical Interpretation of Infinitesimal Strains

- In short,

\[ \varepsilon_{xx} = \varepsilon_x \]
\[ \varepsilon_{yy} = \varepsilon_y \]
\[ \varepsilon_{zz} = \varepsilon_z \]

\[ \varepsilon_{xy} = -\frac{1}{2} \Delta \theta_{xy} \]
\[ \varepsilon_{xz} = -\frac{1}{2} \Delta \theta_{xz} \]
\[ \varepsilon_{yz} = -\frac{1}{2} \Delta \theta_{yz} \]
Engineering Strains

- Using an **engineering notation**, instead of the scientific notation, the components of the infinitesimal strain tensor are

\[
\varepsilon = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}
\end{bmatrix}
\]

\[
\equiv
\begin{bmatrix}
\varepsilon_x & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\
\frac{1}{2} \gamma_{xy} & \varepsilon_y & \frac{1}{2} \gamma_{yz} \\
\frac{1}{2} \gamma_{xz} & \frac{1}{2} \gamma_{yz} & \varepsilon_z
\end{bmatrix}
\]

**REMARK**

Positive longitudinal strains indicate increase in segment length.

Positive angular strains indicate the corresponding angles decrease with the deformation process.

- Because of the symmetry of \( \varepsilon \), the tensor can be written as a 6-component infinitesimal strain vector, (Voigt’s notation):

\[
\varepsilon \in \mathbb{R}^6 \quad \varepsilon \overset{\text{def}}{=} \begin{bmatrix}
\varepsilon_x, & \varepsilon_y, & \varepsilon_z, & \gamma_{xy}, & \gamma_{xz}, & \gamma_{yz}
\end{bmatrix}^T
\]

\[\text{longitudinal strains} \quad \text{angular strains}\]
Variation of Angles

- Consider two segments in the reference configuration with the same origin an angle $\Theta$ between them.

$$\cos \theta = \frac{T^{(1)} \cdot (1 + 2E) \cdot T^{(3)}}{\sqrt{1 + 2T^{(1)} \cdot E \cdot T^{(1)}} \sqrt{1 + 2T^{(2)} \cdot E \cdot T^{(2)}}}$$

$$\theta = \Theta + \Delta \theta$$

$$E = \varepsilon$$

$$\cos(\Theta + \Delta \theta) = \frac{T^{(1)} \cdot [1 + 2\varepsilon] \cdot T^{(2)}}{\sqrt{1 + 2T^{(1)} \cdot \varepsilon \cdot T^{(1)}} \sqrt{1 + 2T^{(2)} \cdot \varepsilon \cdot T^{(2)}}} \approx 1$$

$$\approx 1$$

$$\cos(\Theta + \Delta \theta) = T^{(1)} \cdot T^{(2)} + 2T^{(1)} \cdot \varepsilon \cdot T^{(2)}$$
Variation of Angles

\[ \cos(\Theta + \Delta \theta) = T^{(1)} \cdot T^{(2)} + 2T^{(1)} \cdot \varepsilon \cdot T^{(2)} \]

- \( T^{(1)} \) and \( T^{(2)} \) are unit vectors in the directions of the original segments, therefore, \( T^{(1)} \cdot T^{(2)} = \left\| T^{(1)} \right\| \left\| T^{(2)} \right\| \cos \Theta = \cos \Theta \)

- Also,
  \[ \cos(\Theta + \Delta \theta) = \cos \Theta \cdot \cos \Delta \theta - \sin \Theta \cdot \sin \Delta \theta = \cos \Theta - \sin \Theta \cdot \Delta \theta \]
  \[ \approx 1 \]
  \[ \approx \Delta \theta \]
  \[ \cos \Theta - \sin \Theta \cdot \Delta \theta = \cos \Theta + 2T^{(1)} \cdot \varepsilon \cdot T^{(2)} \]

\[ \Delta \theta = - \frac{2T^{(1)} \cdot \varepsilon \cdot T^{(2)}}{\sin \Theta} = - \frac{2t^{(1)} \cdot \varepsilon \cdot t^{(2)}}{\sin \theta} \]

**REMARK**

The Taylor linear series expansion of \( \sin x \) and \( \cos x \) yield

\[ \sin(x) \approx \sin(0) + \frac{d \sin}{dx} \bigg|_{x=0} (x) + \ldots = x + O(x^2) \]

\[ \cos(x) \approx \cos(0) + \frac{d \cos}{dx} \bigg|_{x=0} (x) + \ldots = 1 + O(x^2) \]
Polar Decomposition

- **Polar decomposition** in finite-strain problems:

\[
\begin{align*}
U &= \sqrt{F^T \cdot F} \\
V &= \sqrt{F \cdot F^T} \\
Q &= F \cdot U^{-1} = V^{-1} \cdot F
\end{align*}
\]

\( \Rightarrow \quad F = Q \cdot U = V \cdot Q \)

**REMARK**

In Infinitesimal Strain Theory \( x \approx X \), therefore, \( F = \frac{\partial x}{\partial X} \approx 1 \)
In Infinitesimal Strain Theory:

\[ \mathbf{U} = \sqrt{\mathbf{F}^T \mathbf{F}} = \sqrt{(\mathbf{I} + \mathbf{J}^T) \cdot (\mathbf{I} + \mathbf{J})} = \sqrt{\mathbf{I} + \mathbf{J} + \mathbf{J}^T + \mathbf{J}^T \cdot \mathbf{J}} \approx \sqrt{\mathbf{I} + \mathbf{J} + \mathbf{J}^T} = \mathbf{I} + \frac{1}{2} (\mathbf{J} + \mathbf{J}^T) \]

\[ \mathbf{U} = \mathbf{I} + \mathbf{\varepsilon} \]

infinitesimal strain tensor

Similarly,

\[ \mathbf{U}^{-1} = (\mathbf{I} + \mathbf{\varepsilon})^{-1} = \mathbf{I} - \mathbf{\varepsilon} = \mathbf{I} - \frac{1}{2} (\mathbf{J} + \mathbf{J}^T) \]

\[ \mathbf{U}^{-1} = \mathbf{I} - \mathbf{\varepsilon} \]

infinitesimal strain tensor

REMARK

The Taylor linear series expansion of \( \sqrt{1 + x} \) and \( (1 + x)^{-1} \) yield

\[ \lambda(x) = \sqrt{1 + x} \cong \lambda(0) + \frac{d\lambda}{dx} \bigg|_{x=0} x = 1 + \frac{1}{2} x + O(x^2) \]

\[ \lambda(x) = (1 + x)^{-1} \cong \lambda(0) + \frac{d\lambda}{dx} \bigg|_{x=0} x = 1 - x + O(x^2) \]
Polar Decomposition

\[ Q = F \cdot U^{-1} = (1 + J) \cdot \left[ 1 - \frac{1}{2} (J + J^T) \right] = 1 + J - \frac{1}{2} (J + J^T) - \frac{1}{2} J \cdot (J + J^T) = 1 + \frac{1}{2} (J - J^T) = \Omega \]

- The infinitesimal rotation tensor \( \Omega \) is defined:

\[
\Omega = \frac{1}{2} (J - J^T) = \frac{1}{2} (u \otimes \nabla - \nabla \otimes u) = \nabla^a u
\]

\[
\Omega_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right] \ll 1 \quad i, j \in \{1, 2, 3\}
\]

- The diagonal terms of \( \Omega \) are zero: \( \Omega = \begin{bmatrix} 0 & \Omega_{12} & -\Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ \Omega_{31} & -\Omega_{23} & 0 \end{bmatrix} \)

- It can be expressed as an infinitesimal rotation vector \( \theta \),

\[
\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -\Omega_{23} \\ -\Omega_{31} \\ -\Omega_{12} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{bmatrix} \ll J
\]

\[
\theta = \frac{1}{2} \nabla \times u
\]

**REMARK**

The antisymmetric or skew-symmetric gradient operator is defined as:

\[
\nabla^a (\bullet) = \frac{1}{2} [\bullet \otimes \nabla - \nabla \otimes \bullet]
\]

**REMARK**

\( \Omega \) is a skew-symmetric tensor and its components are infinitesimal.
From any skew-symmetric tensor $\Omega$, it can be extracted a vector $\theta$ (axial vector of $\Omega$) exhibiting the following property:

$$\Omega \cdot r = \theta \times r \quad \forall r$$

As a consequence:

- The resulting vector is orthogonal to $r$.
- If the components of $\Omega$ are infinitesimal, then $\Omega \cdot r = \theta \times r$ is also infinitesimal.
- The vector $r + \Omega \cdot r = r + \theta \times r$ can be seen as the result of applying a (infinitesimal) rotation (of axial vector $\theta$) on the vector $r$. 

\[ \begin{array}{c}
\theta \\
\hline
\theta \times r = \Omega \cdot r
\end{array} \]
Proof of \( \theta \times r = \Omega \cdot r \quad \forall r \)

- The result of the dot product of the infinitesimal rotation tensor, \( \Omega \), and a generic vector, \( r \), is exactly the same as the result of the cross product of the infinitesimal rotation vector, \( \theta \), and this same vector.

\[
[\Omega] = \begin{bmatrix}
0 & \Omega_{12} & -\Omega_{31} \\
-\Omega_{12} & 0 & \Omega_{23} \\
\Omega_{31} & -\Omega_{23} & 0
\end{bmatrix} \rightarrow \theta \equiv \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{bmatrix} = \begin{bmatrix}
-\Omega_{23} \\
-\Omega_{31} \\
-\Omega_{12}
\end{bmatrix} \Rightarrow \Omega \cdot r = \theta \times r \quad \forall r = \begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix}
\]

- Proof:

\[
\theta \times r = \text{det} \begin{bmatrix}
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
\theta_1 & \theta_2 & \theta_3 \\
r_1 & r_2 & r_3
\end{bmatrix} = \text{det} \begin{bmatrix}
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
-\Omega_{23} & -\Omega_{31} & -\Omega_{12} \\
r_1 & r_2 & r_3
\end{bmatrix} = \begin{bmatrix}
\Omega_{12}r_2 - \Omega_{31}r_3 \\
-\Omega_{12}r_1 + \Omega_{23}r_3 \\
\Omega_{31}r_1 - \Omega_{23}r_2
\end{bmatrix}
\]

\[
\Omega \cdot r = \begin{bmatrix}
0 & \Omega_{12} & -\Omega_{31} \\
-\Omega_{12} & 0 & \Omega_{23} \\
\Omega_{31} & -\Omega_{23} & 0
\end{bmatrix} \begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix} = \begin{bmatrix}
\Omega_{12}r_2 - \Omega_{31}r_3 \\
-\Omega_{12}r_1 + \Omega_{23}r_3 \\
\Omega_{31}r_1 - \Omega_{23}r_2
\end{bmatrix}
\]
Polar Decomposition

- Using:
  \[ J = F - 1 \]
  \[ \varepsilon = \frac{1}{2} \left( J + J^T \right) \]
  \[ Q = 1 + \Omega \]

\[ F = 1 + J = 1 + \frac{1}{2} \left( J + J^T \right) + \frac{1}{2} \left( J - J^T \right) = \varepsilon + \Omega \]

- Consider a differential segment \( dX \):

\[ dx = F \cdot dX = (1 + \varepsilon + \Omega) \cdot dX = \varepsilon \cdot dX + (1 + \Omega) \cdot dX \]

\( F(\bullet) \equiv \text{stretching} (\bullet) + \text{rotation} (\bullet) \)

**REMARK**

The infinitesimal rotation tensor characterizes the rotation and, in the small-strain context, maintains angles and distances.
The volumetric strain:

\[ e = |F| - 1 \]

Considering: \( F = Q \cdot U \) and \( U = 1 + \varepsilon \)

\[
|F| = |Q \cdot U| = |Q| |U| = |U| = |1 + \varepsilon| = \det \begin{vmatrix} 1 + \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & 1 + \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & 1 + \varepsilon_{zz} \end{vmatrix} = \\
1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} + O(\varepsilon^2) \approx 1 + Tr(\varepsilon) \\
= Tr(\varepsilon) \\
\]

\[ e = Tr(\varepsilon) \]
2.13 Strain Rate

Ch.2. Deformation and Strain

REMARK
We are no longer assuming an infinitesimal strain framework
Consider the relative velocity between two points in space at a given (current) instant:

\[
\begin{align*}
\mathbf{v}_{p'} &= \mathbf{v}(x, t) = \mathbf{v}(x_1, x_2, x_3, t) \\
\mathbf{d}v(x, t) &= \mathbf{v}_{q'} - \mathbf{v}_{p'} = \mathbf{v}(\mathbf{x} + \mathbf{d}\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)
\end{align*}
\]

\[
\mathbf{d}v = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot d\mathbf{x} = \mathbf{l} \cdot d\mathbf{x}
\]

\[
dv_i = \frac{\partial v_i}{\partial x_j} \, dx_j = l_{ij} \, dx_j \quad i, j \in \{1, 2, 3\}
\]

\[
\begin{align*}
\mathbf{l}(x, t) &\stackrel{\text{def}}{=} \frac{\partial \mathbf{v}(x, t)}{\partial \mathbf{x}} = \mathbf{v} \otimes \nabla \\
l_{ij} &= \frac{\partial v_i}{\partial x_j} \quad i, j \in \{1, 2, 3\}
\end{align*}
\]
The spatial velocity gradient tensor can be split into a symmetrical and a skew-symmetrical tensor:

\[
\begin{align*}
\mathbf{l} &= \mathbf{v} \otimes \nabla \\
\mathbf{l}_{ij} &= \frac{\partial v_i}{\partial x_j} \quad i, j \in \{1, 2, 3\}
\end{align*}
\]

\[
\mathbf{I} = \text{sym} \mathbf{[l]} + \text{skew} \mathbf{[l]} =: \mathbf{d} + \mathbf{w}
\]

**Strain Rate Tensor**

\[
\mathbf{d} = \text{sym} \mathbf{[l]} = \frac{1}{2} (\mathbf{l} + \mathbf{l}^T) = \frac{1}{2} (\mathbf{v} \otimes \nabla + \nabla \otimes \mathbf{v})^{\text{not}} = \nabla^s \mathbf{v}
\]

\[
d_{ij} = \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] \quad i, j \in \{1, 2, 3\}
\]

\[
\begin{bmatrix}
d_{11} & d_{12} & d_{31} \\
d_{12} & d_{22} & d_{23} \\
d_{31} & d_{23} & d_{33}
\end{bmatrix}
\]

**Rotation Rate or Spin Tensor**

\[
w = \text{skew} \mathbf{[l]} = \frac{1}{2} (\mathbf{l} - \mathbf{l}^T) = \frac{1}{2} (\mathbf{v} \otimes \nabla - \nabla \otimes \mathbf{v})^{\text{not}} = \nabla^w \mathbf{v}
\]

\[
w_{ij} = \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right] \quad i, j \in \{1, 2, 3\}
\]

\[
\begin{bmatrix}
0 & w_{12} & -w_{31} \\
-w_{12} & 0 & w_{23} \\
w_{31} & -w_{23} & 0
\end{bmatrix}
\]
Physical Interpretation of $d$

- The **strain rate** measures the rate of deformation of the square of the differential length $ds$ in the spatial configuration,

$$\frac{d}{dt}(ds(t))^2 = \frac{d}{dt}(dx \cdot dx) = \frac{d}{dt}(dx) \cdot dx + dx \cdot \frac{d}{dt}(dx) = d\left(\frac{dx}{dt}\right)dx + dx \cdot d\left(\frac{dx}{dt}\right) = dv \cdot dx + dx \cdot dv$$

- $$d = \frac{1}{2}(l + l^T)$$

- $$\frac{d}{dt}(ds(t))^2 = \left(dx \cdot l^T\right)dx + dx \cdot \left(l \cdot dx\right) = dx \cdot \left[l^T + l\right]dx = 2dx \cdot d \cdot dx$$

- Differentiating w.r.t. time the expression $(ds(t))^2 - (dS)^2 = 2 dX \cdot E \cdot dX = $

$$\frac{d}{dt}\left(\frac{(ds(t))^2 - (dS)^2}{constant}\right) = \frac{d}{dt}\left(2dX \cdot E(X,t) \cdot dX\right) = 2dX \cdot \frac{dE}{dt} \cdot dX = \frac{d}{dt}\left((ds(t))^2\right)$$

$$= \dot{E}$$
Physical Interpretation of \( \mathbf{d} \)

\[
d\mathbf{X} \cdot \dot{\mathbf{E}} \cdot d\mathbf{X} = dx \cdot d \cdot dx
\]

\[
dx = F \cdot d\mathbf{X}
\]

\[
d\mathbf{X} \cdot \dot{\mathbf{E}} \cdot d\mathbf{X} = dx \cdot d \cdot dx = \left[ \frac{dx}{F \cdot dx} \right]^T \left[ d \right] \left[ \frac{dx}{F \cdot dx} \right] = \left[ \frac{F \cdot d\mathbf{X}}{d\mathbf{X}} \right]^T \left[ d \right] \left[ F \cdot d\mathbf{X} \right] = d\mathbf{X} \cdot (F^T \cdot d \cdot F) \cdot d\mathbf{X}
\]

- And, rearranging terms:

\[
d\mathbf{X} \cdot \left[ F^T \cdot d \cdot F - \dot{\mathbf{E}} \right] \cdot d\mathbf{X} = 0 \quad \forall d\mathbf{X} \quad \rightarrow \quad \left[ F^T \cdot d \cdot F - \dot{\mathbf{E}} \right] = 0 \quad \rightarrow \quad \dot{\mathbf{E}} = F^T \cdot d \cdot F
\]

- There is a direct relation between the material derivative of the material strain tensor and the strain rate tensor but they are not the same.

- \( \dot{\mathbf{E}} \) and \( \mathbf{d} \) will coincide when in the reference configuration \( F \big|_{t=t_0} = 1 \).

**REMARK**

Given a 2\textsuperscript{nd} order tensor \( \mathbf{A} \), if \( x \cdot A \cdot x = 0 \) for any vector \( x \neq 0 \) then \( A = 0 \).
Physical Interpretation of $\mathbf{w}$

- To determine the (skew-symmetric) rotation rate (spin) tensor only three different components are needed:

$$w_{ij} = \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right] \quad i, j \in \{1, 2, 3\}$$

$$\begin{bmatrix} w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{bmatrix}$$

- The spin vector (axial vector $[\mathbf{w}]$) of can be extracted:

$$\omega = \frac{1}{2} \text{rot}(\mathbf{v}) = \frac{1}{2} \nabla \times \mathbf{v} = \frac{1}{2} \begin{bmatrix} \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \end{bmatrix} = \begin{bmatrix} -w_{23} \\ w_{13} \\ -w_{12} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

- The vector $2\omega = \nabla \times \mathbf{v}$ is named vorticity vector.
Physical Interpretation of $\mathbf{w}$

- It can be proven that the equality $\mathbf{\omega} \times \mathbf{r} = \mathbf{w} \cdot \mathbf{r}$ holds true. Therefore:
  - $\mathbf{\omega}$ is the angular velocity of a rotation movement.
  - $\mathbf{\omega} \times \mathbf{r} = \mathbf{w} \cdot \mathbf{r}$ is the rotation velocity of the point that has $\mathbf{r}$ as its position vector w.r.t. the rotation centre.

- Consider now the relative velocity $d\mathbf{v}$,

$$d\mathbf{v} = \mathbf{d} \cdot dx$$
$$\mathbf{d} = \mathbf{d} + \mathbf{w}$$

$$d\mathbf{v} = \mathbf{d} \cdot dx + \mathbf{w} \cdot dx$$
2.14 Material time Derivatives

Ch.2. Deformation and Strain
Deformation Gradient Tensor $\mathbf{F}$

The material time derivative of the deformation gradient tensor,

\[ F_{ij}(\mathbf{X},t) = \frac{\partial x_i(\mathbf{X},t)}{\partial X_j} \quad i, j \in \{1, 2, 3\} \]

\[ \frac{d}{dt} \mathbf{F} = \dot{\mathbf{F}} = \mathbf{l} \cdot \mathbf{F} \]

\[ \frac{dF_{ij}}{dt} = \dot{F}_{ij} = l_{ik} F_{kj} \quad i, j \in \{1, 2, 3\} \]

\[ \frac{d}{dt} \mathbf{F}_{ij} = \dot{\mathbf{F}}_{ij} = l_{ik} F_{kj} \]

**REMARK**

The equality of cross derivatives applies here:

\[ \frac{\partial^2 (\bullet)}{\partial \mu_i \partial \mu_j} = \frac{\partial^2 (\bullet)}{\partial \mu_j \partial \mu_i} \]
The material time derivative of the inverse deformation gradient tensor,\[ \mathbf{F} \cdot \mathbf{F}^{-1} = 1 \]

\[
\frac{d}{dt}(\mathbf{F} \cdot \mathbf{F}^{-1}) = \frac{d\mathbf{F}}{dt} \cdot \mathbf{F}^{-1} + \mathbf{F} \cdot \frac{d(\mathbf{F}^{-1})}{dt} = 0
\]

\[
\Rightarrow \quad \mathbf{F} \cdot \frac{d(\mathbf{F}^{-1})}{dt} = - \frac{d\mathbf{F}}{dt} \cdot \mathbf{F}^{-1} = -\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}
\]

Rearranging terms,
\[
\frac{d(\mathbf{F}^{-1})}{dt} = -\mathbf{F}^{-1} \cdot \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = -\mathbf{F}^{-1} \cdot \mathbf{l} \cdot \mathbf{F} \cdot \mathbf{F}^{-1} = -\mathbf{F}^{-1} \cdot \mathbf{l}
\]

\[
\frac{d(\mathbf{F}^{-1})}{dt} = -\mathbf{F}^{-1} \cdot \mathbf{l} \quad \Rightarrow \quad \frac{d}{dt} = -\mathbf{F}^{-1} \cdot \mathbf{l}
\]

\[
\frac{d\mathbf{F}^{-1}}{dt} = -\mathbf{F}^{-1} \mathbf{l}_{ij} \quad i, j \in \{1, 2, 3\}
\]
The material time derivative of the material strain tensor has already been derived for the physical interpretation of the deformation rate tensor:

\[ \dot{E} = F^T \cdot d \cdot F \]

A more direct procedure yields the same result:

\[ E = \frac{1}{2} \left( F^T \cdot F - I \right) \]

\[ \frac{dE}{dt} = \dot{E} = \frac{1}{2} \left( \dot{F}^T \cdot F + F^T \cdot \dot{F} \right) = \frac{1}{2} \left( F^T \cdot \dot{\ell}^T \cdot F + F^T \cdot \ell^T \cdot F \right) = F^T \cdot \frac{1}{2} \left( \ell + \ell^T \right) \cdot F = F^T \cdot d \cdot F \]
The material time derivative of the spatial strain tensor,\[ e = \frac{1}{2} \left( I - F^{-T} \cdot F^{-1} \right) \]

\[
\frac{d}{dt} e = \dot{e} = -\frac{1}{2} \left( \dot{F}^{-T} \cdot F^{-1} + F^{-T} \cdot \dot{F}^{-1} \right) = \frac{1}{2} \left( l^T \cdot F^{-T} \cdot F^{-1} + l^T \cdot F^{-T} \cdot F^{-1} \cdot l \right)
\]

\[
\dot{e} = \frac{1}{2} \left( l^T \cdot F^{-T} \cdot F^{-1} + F^{-T} \cdot F^{-1} \cdot l \right)
\]
The material time derivative of the **volume differential** associated to a given particle,

\[ dV(t) = \frac{\partial |F(X,t)|}{\partial t} dV_0(X) \]

\[
\frac{d}{dt} dV(t) = \frac{d}{dt} |F| dV_0 = \frac{d}{dt} |F| = dV
\]

\[
\frac{d}{dt} \left( dV(x,t) \right) = \nabla \cdot v(x,t) dV(x,t)
\]

The material time derivative of the determinant of the **deformation gradient tensor** is:

For a 2\(^{nd}\) order tensor \( A \):

\[
\frac{d |A|}{dA_{ij}} = \frac{d |A|}{dA_{ij}} = |A| \cdot A^{-1}_{ji}
\]

\[
\frac{d |F|}{dF_{ij}} = \frac{d |F|}{dF_{ij}} = |F| \cdot F_{ji}^{-1} \frac{dF_{ji}}{dt} = |F| \cdot F_{kj}^{-1} \frac{dF_{ji}}{dt} \frac{dF_{kj}}{dt} l_{ik}
\]

\[
\frac{d |F|}{dt} = \frac{d |F|}{dF_{ij}} \frac{dF_{ij}}{dt} = \frac{d |F|}{dF_{ij}} \frac{dF_{ij}}{dt} \frac{dF_{kj}}{dt} l_{ik}
\]

\[
\Rightarrow \frac{d |F|}{dt} = |F| \cdot (\nabla \cdot v) = (\nabla \cdot v) |F|
\]
Area differential vector $da$

- The material time derivative of the area differential associated to a given particle,

\[
\frac{d}{dt} da(X,t) = \frac{d}{dt} \left( \frac{d|F|(X,t) \cdot dA(X) \cdot F^{-1}(X,t)}{dt} \right) = \frac{d}{dt} \left( F \cdot dA \cdot F^{-1} \right)
\]

\[
\frac{d}{dt} d\mathbf{a}(t) = \left( \nabla \cdot \mathbf{v} \right) F \cdot dA \cdot F^{-1} = d\mathbf{a}
\]

\[
\frac{d}{dt} \left( d\mathbf{a} \right) = \mathbf{1} \cdot \left( \nabla \cdot \mathbf{v} \right) - d\mathbf{a} \cdot \mathbf{l} = d\mathbf{a} \cdot (\mathbf{1} \cdot \nabla \cdot \mathbf{v} - \mathbf{l})
\]
2.15 Other Coordinate Systems

Ch.2. Deformation and Strain
A curvilinear coordinate system is defined by:

- The coordinates, generically named \( \{a, b, c\} \)
- Its vector basis, \( \{\hat{e}_a, \hat{e}_b, \hat{e}_c\} \), formed by unit vectors \( \|\hat{e}_a\| = \|\hat{e}_b\| = \|\hat{e}_c\| = 1 \).
- If the elements of the basis are orthogonal is is called an orthogonal coordinate system: \( \hat{e}_a \cdot \hat{e}_b = \hat{e}_a \cdot \hat{e}_c = \hat{e}_b \cdot \hat{e}_c = 0 \)

The orientation of the curvilinear basis may change at each point in space, \( \hat{e}_m \equiv \hat{e}_m(\mathbf{x}) \) \( m \in \{a, b, c\} \).

**REMARK**

A curvilinear orthogonal coordinate system can be seen as a mobile Cartesian coordinate system \( \{x', y', z'\} \), associated to a curvilinear basis \( \{\hat{e}_a, \hat{e}_b, \hat{e}_c\} \).
A curvilinear orthogonal coordinate system can be seen as a mobile Cartesian coordinate system \( \{ \mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c \} \), associated to a curvilinear basis \( \{ x', y', z' \} \).

- The components of a vector and a tensor magnitude in the curvilinear orthogonal basis will correspond to those in the given Cartesian local system:

\[
\mathbf{v} = \begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix} = \begin{pmatrix} v_{x'} \\ v_{y'} \\ v_{z'} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} T_{aa} & T_{ab} & T_{ac} \\ T_{ba} & T_{bb} & T_{bc} \\ T_{ca} & T_{cb} & T_{cc} \end{pmatrix} = \begin{pmatrix} T_{x'x'} & T_{x'y'} & T_{x'z'} \\ T_{y'x'} & T_{y'y'} & T_{y'z'} \\ T_{z'x'} & T_{z'y'} & T_{z'z'} \end{pmatrix}
\]

- The components of the curvilinear operators will not be the same as those in the given Cartesian local system.

- They must be obtained for each specific case.
Cylindrical Coordinate System

\[ x(r, \theta, z) \equiv \begin{cases} 
  x = r \cos \theta \\
  y = r \sin \theta \\
  z = z 
\end{cases} \]

\[ dV = r \, d\theta \, dr \, dz \]
Cylindrical Coordinate System

- Nabla operator

\[ \nabla = \frac{\partial}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{e}_\theta + \frac{\partial}{\partial z} \hat{e}_z \quad \Rightarrow \quad \nabla = \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{bmatrix} \]

- Displacement vector

\[ \mathbf{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z \quad \Rightarrow \quad \mathbf{u} = \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} \]

- Velocity vector

\[ \mathbf{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \quad \Rightarrow \quad \mathbf{u} = \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} \]

\[ \mathbf{x}(r, \theta, z) = \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \]
Cylindrical Coordinate System

- Infinitesimal strain tensor

\[ \mathbf{\varepsilon} = \frac{1}{2} \left\{ [\mathbf{u} \otimes \nabla] + [\mathbf{u} \otimes \nabla]^T \right\} = \begin{bmatrix} \varepsilon_{xx}' & \varepsilon_{xy}' & \varepsilon_{xz}' \\ \varepsilon_{xy}' & \varepsilon_{yy}' & \varepsilon_{yz}' \\ \varepsilon_{xz}' & \varepsilon_{yz}' & \varepsilon_{zz}' \end{bmatrix} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{rz} & \varepsilon_{\theta z} & \varepsilon_{zz} \end{bmatrix} \]

\[
\begin{align*}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\varepsilon_{r\theta} &= \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} \right) \\
\varepsilon_{rz} &= \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \\
\varepsilon_{\theta z} &= \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right)
\end{align*}
\]

\[ x(r, \theta, z) = \begin{cases} 
\begin{align*}
 x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\end{cases} \]
Cylindrical Coordinate System

- Strain rate tensor

\[
\mathbf{d} = \frac{1}{2} \left\{ \left[ \mathbf{v} \otimes \nabla \right] + \left[ \mathbf{v} \otimes \nabla \right]^T \right\} = \begin{bmatrix}
    d_{xx'} & d_{xy'} & d_{xz'} \\
    d_{xy'} & d_{yy'} & d_{yz'} \\
    d_{xz'} & d_{yz'} & d_{zz'} \\
\end{bmatrix} = \begin{bmatrix}
    d_{rr} & d_{r\theta} & d_{rz} \\
    d_{r\theta} & d_{\theta\theta} & d_{\theta z} \\
    d_{rz} & d_{\theta z} & d_{zz} \\
\end{bmatrix}
\]

\[
\begin{align*}
    d_{rr} &= \frac{\partial v_r}{\partial r} \\
    d_{\theta\theta} &= \frac{1}{r} \frac{\partial v_{\theta}}{\partial \theta} + \frac{v_r}{r} \\
    d_{zz} &= \frac{\partial v_z}{\partial z}
\end{align*}
\]

\[
\begin{align*}
    d_{r\theta} &= \frac{1}{2} \left( \frac{\partial v_r}{\partial \theta} + \frac{v_{\theta}}{r} \right) - \frac{1}{2} \frac{\partial v_{\theta}}{\partial r} \\
    d_{rz} &= \frac{1}{2} \left( \frac{\partial v_r}{\partial z} + 1 \frac{\partial v_z}{\partial r} \right) \\
    d_{\theta z} &= \frac{1}{2} \left( \frac{\partial v_{\theta}}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)
\end{align*}
\]

\[
x(r, \theta, z) = \begin{cases}
    x = r \cos \theta \\
    y = r \sin \theta \\
    z = z
\end{cases}
\]
Spherical Coordinate System

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]

\[ dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \]
Spherical Coordinate System

- Nabla operator
  \[ \nabla = \frac{\partial}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{e}_\phi \Rightarrow \nabla \equiv \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \end{bmatrix} \]

- Displacement vector
  \[ \mathbf{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_\phi \hat{e}_\phi \Rightarrow \mathbf{u} = \begin{bmatrix} u_r \\ u_\theta \\ u_\phi \end{bmatrix} \]

- Velocity vector
  \[ \mathbf{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_\phi \hat{e}_\phi \Rightarrow \mathbf{u} = \begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix} \]

- Position vector
  \[ \mathbf{x} = \mathbf{x}(r, \theta, \phi) \equiv \begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \]
Spherical Coordinate System

- Infinitesimal strain tensor

\[ \varepsilon = \frac{1}{2} \begin{bmatrix} [\mathbf{u} \otimes \nabla] + [\mathbf{u} \otimes \nabla]^T \end{bmatrix} = \begin{bmatrix} \varepsilon_{x'x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{y'x'} & \varepsilon_{y'y'} & \varepsilon_{y'z'} \\ \varepsilon_{z'x'} & \varepsilon_{z'y'} & \varepsilon_{z'z'} \end{bmatrix} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{r\phi} \\ \varepsilon_{\theta r} & \varepsilon_{\theta \theta} & \varepsilon_{\theta \phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi \theta} & \varepsilon_{\phi \phi} \end{bmatrix} \]

\[ \begin{align*}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\theta \theta} &= \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) \\
\varepsilon_{\phi \phi} &= \frac{1}{r \sin \theta} \left( \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} \cot \phi \right) \\
\varepsilon_{r\theta} &= \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right] \\
\varepsilon_{r\phi} &= \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right] \\
\varepsilon_{\theta \phi} &= \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{r} \cot \phi \right] 
\end{align*} \]

\[ \mathbf{x} = \mathbf{x}(r, \theta, \phi) \equiv \begin{bmatrix} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{bmatrix} \]
Spherical Coordinate System

- **Deformation rate tensor**
  \[
  \mathbf{d} = \frac{1}{2} \left\{ [\mathbf{v} \otimes \nabla] + [\mathbf{v} \otimes \nabla]^T \right\} = \begin{bmatrix}
  d_{xx'} & d_{xy'} & d_{xz'} \\
  d_{xy'} & d_{yy'} & d_{yz'} \\
  d_{xz'} & d_{yz'} & d_{zz'}
\end{bmatrix} = \begin{bmatrix}
  d_{rr} & d_{r\theta} & d_{r\phi} \\
  d_{r\theta} & d_{\theta\theta} & d_{\theta\phi} \\
  d_{r\phi} & d_{\theta\phi} & d_{\phi\phi}
\end{bmatrix}
\]

\[
\begin{aligned}
  d_{rr} &= \frac{\partial v_r}{\partial r} \\
  d_{\theta\theta} &= \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \\
  d_{\phi\phi} &= \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} \cot \phi + \frac{v_\theta}{r} \\
  d_{r\theta} &= \frac{1}{2} \left[ \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} \right] \\
  d_{r\phi} &= \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} \right] \\
  d_{\theta\phi} &= \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} - \frac{v_\phi}{r} \cot \phi \right]
\end{aligned}
\]

- **Position vector**
  \[
  \mathbf{x} = \mathbf{x}(r, \theta, \phi) \equiv \begin{cases}
  x = r \sin \theta \cos \phi \\
  y = r \sin \theta \sin \phi \\
  z = r \cos \theta
\end{cases}
\]
Chapter 2
Strain

2.1 Introduction

Definition 2.1. In the broader context, the concept of deformation no longer refers to the study of the absolute motion of the particles as seen in Chapter 1, but to the study of the relative motion, with respect to a given particle, of the particles in its differential neighborhood.

2.2 Deformation Gradient Tensor

Consider the continuous medium in motion of Figure 2.1. A particle $P$ in the reference configuration $\Omega_0$ occupies the point in space $P'$ in the present configuration $\Omega_t$, and a particle $Q$ situated in the differential neighborhood of $P$ has relative positions with respect to this particle in the reference and present times given by $dX$ and $dx$, respectively. The equation of motion is given by

$$\begin{cases} x = \varphi(X,t) \\ x_i = \varphi_i(X_1,X_2,X_3,t) \end{cases} \quad i \in \{1,2,3\} \quad \text{(2.1)}$$

Differentiating (2.1) with respect to the material coordinates $X$ results in the

Fundamental equation of deformation

$$\begin{cases} dx = F \cdot dX \\ dx_i = \frac{\partial x_i}{\partial X_j} dX_j = F_{ij} dX_j \quad i,j \in \{1,2,3\} \quad \text{(2.2)} \end{cases}$$
Figure 2.1: Continuous medium in motion.

Equation (2.2) defines the material deformation gradient tensor \( \mathbf{F}(\mathbf{X}, t) \)^1.

\[
\text{Material deformation gradient tensor} \quad \mathbf{F} = \mathbf{x} \otimes \nabla
\]

\[
F_{ij} = \frac{\partial x_i}{\partial X_j} \quad i, j \in \{1, 2, 3\}
\]

The explicit components of tensor \( \mathbf{F} \) are given by

\[
\mathbf{F} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} \\ \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} \\ \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} \end{bmatrix}
\]

**Remark 2.1.** The deformation gradient tensor \( \mathbf{F}(\mathbf{X}, t) \) contains the information of the relative motion, along time \( t \), of all the material particles in the differential neighborhood of a given particle, identified by its material coordinates \( \mathbf{X} \). In effect, equation (2.2) provides the evolution of the relative position vector \( d\mathbf{x} \) in terms of the corresponding relative position in the reference time, \( d\mathbf{X} \). Thus, if the value of \( \mathbf{F}(\mathbf{X}, t) \) is known, the information associated with the general concept of deformation defined in Section 2.1 is also known.

---

1 Here, the symbolic form of the material Nabla operator, \( \nabla \equiv \partial_{i_j}/\partial X_i \), applied to the expression of the open or tensor product, \( [a \otimes b]_{ij} = [a b]_{ij} = a_i b_j \), is considered.
2.2.1 Inverse Deformation Gradient Tensor

Consider now the inverse equation of motion

\[
\begin{align*}
X &= \varphi^{-1}(x,t) \not= X(x,t), \\
X_i &= \varphi_i^{-1}(x_1,x_2,x_3,t) \not= X_i(x_1,x_2,x_3,t) \quad i \in \{1,2,3\}.
\end{align*}
\] (2.5)

Differentiating (2.5) with respect to the spatial coordinates \(x_i\) results in

\[
\begin{align*}
dX &= F^{-1} \cdot dx, \\
dX_i &= \frac{\partial X_i}{\partial x_j} dx_j = F^{-1}_{ij} dx_j \quad i,j \in \{1,2,3\}.
\end{align*}
\] (2.6)

The tensor defined in (2.6) is named \textit{spatial deformation gradient tensor} or \textit{inverse (material) deformation gradient tensor} and is characterized by\(^2\)

\[
\begin{align*}
\text{Spatial deformation gradient tensor} \\
F^{-1} \not= X \otimes \nabla \\
F^{-1}_{ij} = \frac{\partial X_i}{\partial x_j} i,j \in \{1,2,3\}
\end{align*}
\] (2.7)

\textit{Remark 2.2.} The spatial deformation gradient tensor, denoted in (2.6) and (2.7) as \(F^{-1}\), is in effect the inverse of the (material) deformation gradient tensor \(F\). The verification is immediate since\(^3\)

\[
\begin{align*}
\frac{\partial x_i}{\partial X_k} \frac{\partial X_k}{\partial x_j} \delta_{ij} &= F_{ik} F^{-1}_{kj}, \\
\frac{\partial x_i}{\partial X_k} \frac{\partial X_k}{\partial x_j} &= F^{-1}_{ik} \delta_{ij} \quad \Rightarrow \quad F \cdot F^{-1} = 1, \label{eq:1}\\
\frac{\partial X_i}{\partial x_k} \frac{\partial x_k}{\partial X_j} &= F^{-1}_{ik} F_{kj} \quad \Rightarrow \quad F^{-1} \cdot F = 1.
\end{align*}
\]

\(^2\) Here, the symbolic form of the \textit{spatial Nabla operator}, \(\nabla \equiv \partial \hat{e}_i / \partial x_i\), is considered. Note the difference in notation between this spatial operator \(\nabla\) and the \textit{material Nabla} \(\nabla\).

\(^3\) The two-index operator \textit{Delta Kronecker} \(\delta_{ij}\) is defined as \(\delta_{ij} = 1\) if \(i = j\) and \(\delta_{ij} = 0\) if \(i \not= j\). The second-order unit tensor \(I\) is given by \(I_{ij} = \delta_{ij}\).
The explicit components of tensor $F^{-1}$ are given by

\[
[F^{-1}] = [X \otimes \nabla] = \left[ \begin{array}{c}
X_1 \\
X_2 \\
X_3
\end{array} \right] \sim \left[ \begin{array}{ccc}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}
\end{array} \right] = \left[ \begin{array}{ccc}
\frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\
\frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\
\frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3}
\end{array} \right].
\]  

(2.8)

**Example 2.1** – At a given time, the motion of a continuous medium is defined by

\[
\begin{aligned}
&x_1 = X_1 - AX_3 \\
&x_2 = X_2 - AX_3 \\
&x_3 = -AX_1 + AX_2 + X_3
\end{aligned}
\]

Obtain the material deformation gradient tensor $F(X,t)$ at this time. By means of the inverse equation of motion, obtain the spatial deformation gradient tensor $F^{-1}(X)$. Using the results obtained, verify that $F \cdot F^{-1} = 1$.

**Solution**

The material deformation gradient tensor is

\[
F = x \otimes \nabla \equiv [X] \nabla = \left[ \begin{array}{c}
X_1 - AX_3 \\
X_2 - AX_3 \\
-AX_1 + AX_2 + X_3
\end{array} \right] \sim \left[ \begin{array}{ccc}
\frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3}
\end{array} \right] = \left[ \begin{array}{ccc}
1 & 0 & -A \\
0 & 1 & -A \\
-A & A & 1
\end{array} \right].
\]

The inverse equation of motion is obtained directly from the algebraic inversion of the equation of motion, \(X(X,t)\),

\[
\begin{aligned}
X_1 &= (1 + A^2)x_1 - A^2x_2 + Ax_3 \\
X_2 &= A^2x_1 + (1 - A^2)x_2 + Ax_3 \\
X_3 &= Ax_1 - Ax_2 + x_3
\end{aligned}
\]
Then, the spatial deformation gradient tensor is

\[
F^{-1} = X \otimes \nabla \equiv [X] [\nabla]^T = \begin{bmatrix}
(1 + A^2) x_1 - A^2 x_2 + A x_3 \\
A^2 x_1 + (1 - A^2) x_2 + A x_3 \\
A x_1 - A x_2 + x_3
\end{bmatrix} \begin{bmatrix}
\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}
\end{bmatrix}
\]

\[
F^{-1} \equiv \begin{bmatrix}
1 + A^2 & -A^2 & A \\
A^2 & 1 - A^2 & A \\
A & -A & 1
\end{bmatrix}.
\]

Finally, it is verified that

\[
F \cdot F^{-1} \equiv \begin{bmatrix}
1 & 0 & -A \\
0 & 1 & -A \\
-A & A & 1
\end{bmatrix} \begin{bmatrix}
1 + A^2 & -A^2 & A \\
A^2 & 1 - A^2 & A \\
A & -A & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \equiv \mathbf{1}.
\]

### 2.3 Displacements

**Definition 2.2.** A displacement is the difference between the position vectors in the present and reference configurations of a same particle.

The displacement of a particle \( P \) at a given time is defined by vector \( \mathbf{u} \), which joins the points in space \( P \) (initial position) and \( P' \) (position at the present time \( t \)) of the particle (see Figure 2.2). The displacement of all the particles in the continuous medium defines a displacement vector field which, as all properties of the continuous medium, can be described in material form \( \mathbf{U}(X,t) \) or in spatial form \( \mathbf{u}(x,t) \) as follows.

\[
\begin{cases}
\mathbf{U}(X,t) = \mathbf{x}(X,t) - \mathbf{X} \\
U_i(X,t) = x_i(X,t) - X_i \\
\end{cases} \quad i \in \{1,2,3\}
\]

\[
\begin{cases}
\mathbf{u}(x,t) = \mathbf{x} - \mathbf{X}(x,t) \\
u_i(x,t) = x_i - X_i(x,t) \\
\end{cases} \quad i \in \{1,2,3\}
\]
2.3.1 Material and Spatial Displacement Gradient Tensors

Differentiation with respect to the material coordinates of the displacement vector $U_i$ defined in (2.9) results in

$$\frac{\partial U_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} \frac{\partial X_i}{\partial x_j} = F_{ij} - \delta_{ij} = J_{ij},$$

(2.11)

which defines the material displacement gradient tensor as follows.

$$J_{ij}(X,t) = U(X,t) \otimes \nabla = F - 1$$

(2.12)

Similarly, differentiation with respect to the spatial coordinates of the expression of $u_i$ given in (2.10) yields

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} \frac{\partial X_i}{\partial x_j} - \delta_{ij} = F_{ij}^{-1} - \delta_{ij},$$

(2.13)
which defines the *spatial displacement gradient tensor* as follows.

\[
\text{Spatial displacement gradient tensor}
\begin{align*}
    j(x,t) & \overset{def}{=} u(x,t) \otimes \nabla = 1 - F^{-1} \\
    j_{ij} & = \frac{\partial u_i}{\partial x_j} = \delta_{ij} - F_{ij}^{-1} \quad i, j \in \{1, 2, 3\}
\end{align*}
\]

(2.15)

\[
\begin{align*}
    u &= j \cdot dx \\
    du_i &= \frac{\partial u_i}{\partial x_j} dx_j = j_{ij} dx_j \quad i, j \in \{1, 2, 3\}
\end{align*}
\]

(2.16)

### 2.4 Strain Tensors

Consider now a particle of the continuous medium that occupies the point in space \( P \) in the material configuration, and another particle \( Q \) in its differential neighborhood separated a segment \( dX \) (with length \( dS = \sqrt{dX \cdot dX} \)) from the previous particle, being \( dx \) (with length \( ds = \sqrt{dx \cdot dx} \)) its counterpart in the present configuration (see Figure 2.3). Both differential vectors are related through the deformation gradient tensor \( F(X,t) \) by means of equations (2.2) and (2.6),

\[
\begin{align*}
    dx &= F \cdot dX \quad \text{and} \quad dX = F^{-1} \cdot dx, \\
    dx_i &= F_{ij} dX_j \quad \text{and} \quad dX_i = F_{ij}^{-1} dX_j \quad i, j \in \{1, 2, 3\}.
\end{align*}
\]

(2.17)

Then,

\[
\begin{align*}
    (ds)^2 &= dx \cdot dx \overset{nat}{=} [dx]^T [dx] = [F \cdot dX]^T [F \cdot dX] \overset{nat}{=} dX \cdot F^T \cdot F \cdot dx \\
    (ds)^2 &= dx_k dx_k = F_{ki} F_{kj} dX_j = dX_i F_{ki} F_{kj} dX_j = dX_i F_{ik}^{-1} dX_j = dX_i F_{ik}^{-1} F_{kj}^{-1} dX_j = dX_i F_{ik}^{-1} F_{kj}^{-1} dX_j
\end{align*}
\]

(2.18)

or, alternatively\(^4\),

\[
\begin{align*}
    (dS)^2 &= dX \cdot dX \overset{nat}{=} [dX]^T [dX] = [F^{-1} \cdot dX]^T [F^{-1} \cdot dX] \overset{nat}{=} dx \cdot F^{-T} \cdot F^{-1} \cdot dx, \\
    (dS)^2 &= dx_k dx_k = F_{ki}^{-1} dx_i F_{kj}^{-1} dx_j = dx_i F_{ki}^{-1} F_{kj}^{-1} dX_j = dx_i F_{ki}^{-1} F_{kj}^{-1} dX_j = dX_i F_{ik}^{-T} F_{kj}^{-1} dX_j.
\end{align*}
\]

(2.19)

\(^4\) The convention \([\ast]^{nat} = (\ast)^T\) is used.
2.4.1 Material Strain Tensor (Green-Lagrange Strain Tensor)

Subtracting expressions (2.18) and (2.19) results in

\[(ds)^2 - (dS)^2 = dX \cdot F^T \cdot F \cdot dX - dX \cdot dX =
\]
\[= dX \cdot F^T \cdot F \cdot dX - dX \cdot 1 \cdot dX =
\]
\[= dX \cdot (F^T \cdot F - 1) \cdot dX = 2 \cdot dX \cdot E \cdot dX, \tag{2.20}
\]

which implicitly defines the material strain tensor or Green-Lagrange strain tensor as follows.

\[
\begin{align*}
\text{Material} & \quad \text{(Green-Lagrange)} \\
\text{strain tensor} & \\
E(X,t) = & \quad \frac{1}{2} (F^T \cdot F - 1) \\
E_{ij}(X,t) = & \quad \frac{1}{2} (F_{ki} F_{kj} - \delta_{ij}) \quad i, j \in \{1, 2, 3\}
\end{align*}
\]

Remark 2.3. The material strain tensor \( E \) is symmetric. Proof is obtained directly from (2.21), observing that

\[
\begin{align*}
E^T & = \frac{1}{2} (F^T \cdot F - 1)^T = \frac{1}{2} (F^T \cdot (F^T)^T - I^T) = \frac{1}{2} (F^T \cdot F - 1) = E, \\
E_{ij} & = E_{ji} \quad i, j \in \{1, 2, 3\}.
\end{align*}
\]
2.4.2 Spatial Strain Tensor (Almansi Strain Tensor)

Subtracting expressions (2.18) and (2.19) in an alternative form yields

\[(ds)^2 - (dS)^2 = d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{x} \cdot \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} \cdot d\mathbf{x} =
\]
\[= d\mathbf{x} \cdot \mathbf{1} \cdot d\mathbf{x} - d\mathbf{x} \cdot \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} \cdot d\mathbf{x} =
\]
\[= d\mathbf{x} \cdot (\mathbf{1} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) \cdot d\mathbf{x} = 2d\mathbf{x} \cdot \mathbf{e} \cdot d\mathbf{x}, \quad (2.22)
\]

which implicitly defines the *spatial strain tensor* or *Almansi strain tensor* as follows.

\[
\text{Spatial (Almansi) strain tensor }
\begin{align*}
\mathbf{e}(\mathbf{x}, t) &= \frac{1}{2} (\mathbf{1} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) \\
\mathbf{e}_{ij}(\mathbf{x}, t) &= \frac{1}{2} (\delta_{ij} - F_{ki}^{-1} F_{kj}^{-1}) \\
i, j &\in \{1, 2, 3\}.
\end{align*}
\]

(2.23)

**Remark 2.4.** The spatial strain tensor \( \mathbf{e} \) is symmetric. Proof is obtained directly from (2.23), observing that

\[
\mathbf{e}^T = \frac{1}{2} (\mathbf{1} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1})^T = \frac{1}{2} (\mathbf{1}^T - (\mathbf{F}^{-1})^T \cdot (\mathbf{F}^{-T})^T) =
\]

\[
= \frac{1}{2} (\mathbf{1} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) = \mathbf{e},
\]

\[e_{ij} = e_{ji}, \quad i, j \in \{1, 2, 3\}. \]

**Example 2.2** – Obtain the material and spatial strain tensors for the motion in Example 2.1.

**Solution**

The material strain tensor is

\[
\mathbf{E}(\mathbf{X}, t) = \frac{1}{2} (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1}) \equiv \frac{1}{2} \left( \begin{array}{ccc} 1 & 0 & -A \\
0 & 1 & A \\
-A & -A & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & -A \\
0 & 1 & A \\
-A & -A & 1 \end{array} \right) - \left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{array} \right) =
\]

\[
= \frac{1}{2} \left[ \begin{array}{ccc} A^2 & -A^2 & -2A \\
-A^2 & A^2 & 0 \\
-2A & 0 & 2A^2 \end{array} \right]
\]

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and the spatial strain tensor is
\[
e(\mathbf{X},t) = \frac{1}{2} (\mathbf{1} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) = \\
\begin{pmatrix}
2 & -A^2 & -A^2 \\
-A^2 & 1 - A^2 & -A \\
A & A & 1
\end{pmatrix}
\begin{pmatrix}
2 & -A^2 & -A^2 \\
A & 1 - A^2 & -A \\
A & A & 1
\end{pmatrix} \\
\begin{pmatrix}
-3A^2 - 2A^4 & A^2 + 2A^4 & -2A - 2A^3 \\
A^2 + 2A^4 & A^2 - 2A^4 & 2A^3 \\
-2A - 2A^3 & 2A^3 & -2A^2
\end{pmatrix}.
\]

Observe that \( \mathbf{E} \neq \mathbf{e} \).

Remark 2.5. The material strain tensor \( \mathbf{E} \) and the spatial strain tensor \( \mathbf{e} \) are different tensors. They are not the material and spatial descriptions of a same strain tensor. Expressions (2.20) and (2.22),
\[
(ds)^2 - (dS)^2 = 2d\mathbf{X} \cdot \mathbf{E} \cdot d\mathbf{X} = 2d\mathbf{x} \cdot \mathbf{e} \cdot d\mathbf{x},
\]
clearly show this since each tensor is affected by a different vector \((d\mathbf{X} \text{ and } d\mathbf{x}, \text{ respectively})\).

The Green-Lagrange strain tensor is naturally described in material description \((\mathbf{E} (\mathbf{X},t))\). In equation (2.20) it acts on element \(d\mathbf{X}\) (defined in material configuration) and, hence, its denomination as material strain tensor. However, as all properties of the continuous medium, it may be described, if required, in spatial form \((\mathbf{E} (\mathbf{x},t))\) through the adequate substitution of the equation of motion.

The contrary occurs with the Almansi strain tensor: it is naturally described in spatial form and in equation (2.22) acts on the differential vector \(d\mathbf{x}\) (defined in the spatial configuration) and, thus, its denomination as spatial strain tensor. It may also be described, if required, in material form \((\mathbf{e} (\mathbf{X},t))\).
2.4.3 Strain Tensors in terms of the Displacement (Gradients)

Replacing expressions (2.12) and (2.15) into equations (2.21) and (2.23) yields the expressions of the strain tensors in terms of the material displacement gradient, \( \mathbf{J}(\mathbf{X},t) \), and the spatial displacement gradient, \( \mathbf{j}(\mathbf{x},t) \).

\[
\mathbf{E} = \frac{1}{2} \left( (1 + \mathbf{J}^T) \cdot (1 + \mathbf{J}) - 1 \right) = \frac{1}{2} (\mathbf{J} + \mathbf{J}^T + \mathbf{J}^T \cdot \mathbf{J})
\]
\[E_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i} \frac{\partial U_k}{\partial X_j} \right) \quad i, j \in \{1, 2, 3\} \quad (2.24)
\]

\[
e = \frac{1}{2} \left( (1 - (1 - \mathbf{j}^T) \cdot (1 - \mathbf{j}) \right) = \frac{1}{2} (\mathbf{j} + \mathbf{j}^T - \mathbf{j}^T \cdot \mathbf{j})
\]
\[e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) \quad i, j \in \{1, 2, 3\} \quad (2.25)
\]

2.5 Variation of Distances: Stretch and Unit Elongation

Consider now a particle \( P \) in the reference configuration and another particle \( Q \), belonging to the differential neighborhood of \( P \) (see Figure 2.4). The corresponding positions in the present configuration are given by the points in space \( P' \) and \( Q' \) such that the distance between the two particles in the reference configuration, \( dS \), is transformed into \( ds \) at the present time. The vectors \( \mathbf{T} \) and \( \mathbf{t} \) are the unit vectors in the directions \( \overrightarrow{PQ} \) and \( \overrightarrow{P'Q'} \), respectively.

**Definition 2.3.** The **stretch or stretch ratio** of a material point \( P \) (or a spatial point \( P' \)) in the material direction \( \mathbf{T} \) (or spatial direction \( \mathbf{t} \)) is the length of the deformed differential segment \( \overrightarrow{P'Q'} \) per unit of length of the original differential segment \( \overrightarrow{PQ} \).

The translation of the previous definition into mathematical language is

\[
\text{Stretch} \overset{\text{def}}{=} \lambda_T = \lambda_t = \frac{P'Q'}{PQ} = \frac{ds}{dS} \quad (0 < \lambda < \infty) \quad . \quad (2.26)
\]
Definition 2.4. The unit elongation, elongation ratio or extension of a material point $P$ (or a spatial point $P'$) in the material direction $T$ (or spatial direction $t$) is the increment of length of the deformed differential segment $P'Q$ per unit of length of the original differential segment $PQ$.

The corresponding mathematical definition is

\[
\text{Unit elongation} \stackrel{\text{def}}{=} \varepsilon_T = \varepsilon_t = \frac{\Delta PQ}{PQ} = \frac{ds - dS}{dS}.
\] (2.27)

Equations (2.26) and (2.27) allow immediately relating the values of the unit elongation and the stretch for a same point and direction as follows.

\[
\varepsilon = \frac{ds - dS}{dS} = \frac{ds}{dS} - 1 = \lambda - 1 \quad (\Rightarrow -1 < \varepsilon < \infty)
\] (2.28)

Often, the subindices $\bullet_T$ and $\bullet_t$ will be dropped when referring to stretches or unit elongations. However, one must bear in mind that both stretches and unit elongations are always associated with a particular direction.
Remark 2.6. The following deformations may take place:

- If $\lambda = 1$ ($\varepsilon = 0$) $\Rightarrow ds = dS$: The particles $P$ and $Q$ may have moved along time, but without increasing or decreasing the distance between them.
- If $\lambda > 1$ ($\varepsilon > 0$) $\Rightarrow ds > dS$: The distance between the particles $P$ and $Q$ has lengthened with the deformation of the medium.
- If $\lambda < 1$ ($\varepsilon < 0$) $\Rightarrow ds < dS$: The distance between the particles $P$ and $Q$ has shortened with the deformation of the medium.

2.5.1 Stretches, Unit Elongations and Strain Tensors

Consider equations (2.21) and (2.22) as well as the geometric expressions $dX = T \cdot dS$ and $dx = t \cdot ds$ (see Figure 2.4). Then,

\[
\begin{cases}
(ds)^2 - (dS)^2 = 2 \frac{dX}{dS} \cdot E \cdot dX = 2 (dS)^2 T \cdot E \cdot T \\
(ds)^2 - (dS)^2 = 2 \frac{dx}{ds} \cdot t \cdot e \cdot dx = 2 (ds)^2 t \cdot e \cdot t
\end{cases}
\]

(2.29)

and dividing these expressions by $(dS)^2$ and $(ds)^2$, respectively, results in

\[
\left( \frac{ds}{dS} \right)^2 = 1 - \lambda^2 - 1 = 2 T \cdot E \cdot T \Rightarrow \lambda = \sqrt{1 + 2 T \cdot E \cdot T} \\
\varepsilon = \lambda - 1 = \sqrt{1 + 2 T \cdot E \cdot T} - 1
\]

(2.30)

\[
1 - \left( \frac{ds}{ds} \right)^2 = 1 - \left( \frac{1}{\lambda} \right)^2 = 2 t \cdot e \cdot t \Rightarrow \lambda = \frac{1}{\sqrt{1 - 2 t \cdot e \cdot t}} \\
\varepsilon = \lambda - 1 = \frac{1}{\sqrt{1 - 2 t \cdot e \cdot t}} - 1
\]

(2.31)

These equations allow calculating the unit elongation and stretch for a given direction (in material description, $T$, or in spatial description, $t$).
Remark 2.7. The material and spatial strain tensors, $E(X,t)$ and $e(x,t)$, contain information on the stretches (and unit elongations) for any direction in a differential neighborhood of a given particle, as evidenced by (2.30) and (2.31).

Example 2.3 – The spatial strain tensor for a given motion is

$$e(x,t) \equiv \begin{bmatrix} 0 & 0 & -te^{iz} \\ 0 & 0 & 0 \\ -te^{iz} & 0 & t(2e^{iz} - e^{i}) \end{bmatrix}.$$ 

Calculate the length, at time $t = 0$, of the segment that at time $t = 2$ is rectilinear and joins points $a \equiv (0,0,0)$ and $b \equiv (1,1,1)$.

Solution

The shape and geometric position of the material segment at time $t = 2$ is known. At time $t = 0$ (reference time) the segment is not necessarily rectilinear and the positions of its extremes $A$ and $B$ (see figure below) are not known. To determine its length, (2.31) is applied for a unit vector in the direction of the spatial configuration $t$,

$$\lambda = \frac{1}{\sqrt{1 - 2t \cdot e \cdot t}} \frac{ds}{dS} \implies dS = \frac{1}{\lambda} ds .$$
To obtain the stretch in the direction \( t \equiv [1, 1, 1]^T / \sqrt{3} \), the expression \( t \cdot e \cdot t \) is computed first as

\[
t \cdot e \cdot t \equiv \frac{1}{\sqrt{3}} [1, 1, 1] \begin{bmatrix} 0 & 0 & -te^{\ell z} \\ 0 & 0 & 0 \\ -te^{\ell z} & 0 & t(2e^{\ell z} - e') \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} = -\frac{1}{3} te'.
\]

Then, the corresponding stretch at time \( t = 2 \) is

\[
\lambda = \frac{1}{\sqrt{1 + \frac{4}{3} te'}} \implies \lambda \bigg|_{t=2} = \frac{1}{\sqrt{1 + \frac{4}{3} e^2}} = \frac{\sqrt{3}}{\sqrt{3 + 4e^2}}.
\]

The length at time \( t = 0 \) of the segment \( AB \) is

\[
l_{AB} = \int_A^B dS = \int_a^b \frac{1}{\lambda} ds = \frac{1}{\lambda} \sqrt{b-a} = \frac{l_{ab}}{\lambda} = \frac{1}{2} l_{ab} = \frac{1}{2} \sqrt{3}
\]

and replacing the expression obtained above for the stretch at time \( t = 2 \) finally results in

\[
l_{AB} = \sqrt{3 + 4e^2}.
\]

### 2.6 Variation of Angles

Consider a particle \( P \) and two additional particles \( Q \) and \( R \), belonging to the differential neighborhood of \( P \) in the material configuration (see Figure 2.5), and the same particles occupying the spatial positions \( P', Q' \) and \( R' \). The relationship between the angles that form the corresponding differential segments in the reference configuration (angle \( \Theta \)) and the present configuration (angle \( \theta \)) is to be considered next.

Applying (2.2) and (2.6) on the differential vectors that separate the particles,

\[
\begin{align*}
\begin{cases} dx^{(1)} = F \cdot dX^{(1)} \\ dx^{(2)} = F \cdot dX^{(2)} \end{cases} & \implies \begin{cases} dX^{(1)} = F^{-1} \cdot dx^{(1)} \\ dX^{(2)} = F^{-1} \cdot dx^{(2)} \end{cases} \tag{2.32}
\end{align*}
\]

and using the definitions of the unit vectors \( T^{(1)}, T^{(2)}, t^{(1)} \) and \( t^{(2)} \) that establish the corresponding directions in Figure 2.5,
Figure 2.5: Angles between particles in a continuous medium.

\[
\begin{align*}
\frac{dX^{(1)}}{} &= dS^{(1)} T^{(1)}
\frac{dX^{(2)}}{} &= dS^{(2)} T^{(2)}
\Rightarrow \quad \frac{dx^{(1)}}{} &= \lambda^{(1)} ds^{(1)}
\frac{dx^{(2)}}{} &= \lambda^{(2)} ds^{(2)}.
\end{align*}
\] (2.33)

Finally, according to the definition in (2.26), the corresponding stretches are

\[
\begin{align*}
\frac{ds^{(1)}}{} &= \lambda^{(1)} dS^{(1)}
\frac{ds^{(2)}}{} &= \lambda^{(2)} ds^{(2)}
\Rightarrow \quad \frac{dS^{(1)}}{} &= \frac{1}{\lambda^{(1)}} ds^{(1)}
\frac{dS^{(2)}}{} &= \frac{1}{\lambda^{(2)}} ds^{(2)}.
\end{align*}
\] (2.34)

Expanding now the scalar product of the vectors \(dx^{(1)}\) and \(dx^{(2)}\),

\[
\begin{align*}
\frac{ds^{(1)}}{} \cdot \frac{ds^{(2)}}{} \cos \theta &= \left| \frac{dx^{(1)}}{} \right| \left| \frac{dx^{(2)}}{} \right| \cos \theta = \frac{dx^{(1)}}{} \cdot \frac{dx^{(2)}}{},
&= \left( \frac{F \cdot dX^{(1)}}{} \right)^T \left( \frac{F \cdot dX^{(2)}}{} \right) = \frac{dX^{(1)}}{} \cdot (F^T \cdot F) \cdot \frac{dX^{(2)}}{} = \frac{dx^{(1)}}{} \cdot (2E + 1) \cdot \frac{dx^{(2)}}{},
&= \frac{1}{\lambda^{(1)}} dS^{(1)} \cdot (2E + 1) \cdot T^{(2)} = \frac{1}{\lambda^{(2)}} ds^{(2)}.
\end{align*}
\] (2.35)

\(\text{6 The scalar product of two vectors } a \text{ and } b \text{ is defined in terms of the angle between them, } \theta, \text{ as } a \cdot b = |a| \cdot |b| \cos \theta.\)
and, comparing the initial and final terms in (2.35), yields
\[
\cos \theta = \frac{T^{(1)} \cdot (1 + 2E) \cdot T^{(2)}}{\lambda^{(1)} \lambda^{(2)}},
\]
(2.36)

where the stretches \( \lambda^{(1)} \) and \( \lambda^{(2)} \) can be obtained by applying (2.30) to the directions \( T^{(1)} \) and \( T^{(2)} \), resulting in
\[
\cos \theta = \frac{T^{(1)} \cdot (1 + 2E) \cdot T^{(2)}}{\sqrt{1 + 2T^{(1)} \cdot E \cdot T^{(1)}} \sqrt{1 + 2T^{(2)} \cdot E \cdot T^{(2)}}}.
\]
(2.37)

In an analogous way, operating on the reference configuration, the angle \( \Theta \) between the differential segments \( dX^{(1)} \) and \( dX^{(2)} \) (in terms of \( t^{(1)} \), \( t^{(2)} \) and \( e \)) is obtained,
\[
\cos \Theta = \frac{t^{(1)} \cdot (1 - 2e) \cdot t^{(2)}}{\sqrt{1 - 2t^{(1)} \cdot e \cdot t^{(1)}} \sqrt{1 - 2t^{(2)} \cdot e \cdot t^{(2)}}}.
\]
(2.38)

**Remark 2.8.** Similarly to the discussion in Remark 2.7, the material and spatial strain tensors, \( E(X,t) \) and \( e(x,t) \), also contain information on the variation of the angles between differential segments in the differential neighborhood of a particle during the deformation process. These facts will be the basis for providing a physical interpretation of the components of the strain tensors in Section 2.7.

### 2.7 Physical Interpretation of the Strain Tensors

#### 2.7.1 Material Strain Tensor

Consider a segment \( PQ \), oriented parallel to the \( X_1 \)-axis in the reference configuration (see Figure 2.6). Before the deformation takes place, \( PQ \) has a known length \( dS = dX \).

The length of \( P'Q' \) is sought. To this aim, consider the material strain tensor \( E \) given by its components,
\[
E = \begin{bmatrix}
E_{XX} & E_{XY} & E_{XZ} \\
E_{XY} & E_{YY} & E_{YZ} \\
E_{XZ} & E_{YZ} & E_{ZZ}
\end{bmatrix}
= \begin{bmatrix}
E_{11} & E_{12} & E_{13} \\
E_{12} & E_{22} & E_{23} \\
E_{13} & E_{23} & E_{33}
\end{bmatrix}
\]
(2.39)
Consequently,

\[ \mathbf{T} \cdot \mathbf{E} \cdot \mathbf{T} \equiv [\mathbf{T}]^T [\mathbf{E}] [\mathbf{T}] = [1, 0, 0] \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = E_{11}. \] (2.40)

The stretch in the material direction \( X_1 \) is now obtained by replacing the value \( \mathbf{T} \cdot \mathbf{E} \cdot \mathbf{T} \) into the expression for stretch (2.30), resulting in

\[ \lambda_1 = \sqrt{1 + 2E_{11}} \Rightarrow \varepsilon_X = \lambda_X - 1 = \sqrt{1 + 2E_{XX}} - 1 \]

\[ \lambda_2 = \sqrt{1 + 2E_{22}} \Rightarrow \varepsilon_Y = \lambda_Y - 1 = \sqrt{1 + 2E_{YY}} - 1 \]

\[ \lambda_3 = \sqrt{1 + 2E_{33}} \Rightarrow \varepsilon_Z = \lambda_Z - 1 = \sqrt{1 + 2E_{ZZ}} - 1 \] (2.41)

Remark 2.9. The components \( E_{XX}, E_{YY} \) and \( E_{ZZ} \) (or \( E_{11}, E_{22} \) and \( E_{33} \)) of the main diagonal of tensor \( \mathbf{E} \) (denoted longitudinal strains) contain the information on stretch and unit elongations of the differential segments that were initially (in the reference configuration) oriented in the directions \( X, Y \) and \( Z \), respectively.

- If \( E_{XX} = 0 \Rightarrow \varepsilon_X = 0 \) : No unit elongation in direction \( X \).
- If \( E_{YY} = 0 \Rightarrow \varepsilon_Y = 0 \) : No unit elongation in direction \( Y \).
- If \( E_{ZZ} = 0 \Rightarrow \varepsilon_Z = 0 \) : No unit elongation in direction \( Z \).
Consider now the angle between segments $\overline{PQ}$ (parallel to the $X_1$-axis) and $\overline{PR}$ (parallel to the $X_2$-axis), where $Q$ and $R$ are two particles in the differential neighborhood of $P$ in the material configuration and $P'$, $Q'$ and $R'$ are the respective positions in the spatial configuration (see Figure 2.7). If the angle ($\Theta = \pi/2$) between the segments in the reference configuration is known, the angle $\theta$ in the present configuration can be determined using $\Theta$ and taking into account their orthogonality ($\mathbf{T}^{(1)} \cdot \mathbf{T}^{(2)} = 0$) and the equalities $\mathbf{T}^{(1)} \cdot \mathbf{E} \cdot \mathbf{T}^{(1)} = E_{11}$, $\mathbf{T}^{(2)} \cdot \mathbf{E} \cdot \mathbf{T}^{(2)} = E_{22}$ and $\mathbf{T}^{(1)} \cdot \mathbf{E} \cdot \mathbf{T}^{(2)} = E_{12}$. That is,

$$
\cos \theta = \frac{\mathbf{T}^{(1)} \cdot (1 + 2\mathbf{E}) \cdot \mathbf{T}^{(2)}}{\sqrt{1 + 2E_{11}} \sqrt{1 + 2E_{22}}} \left(1 + 2E_{11} \sqrt{1 + 2E_{22}} \right),
$$

(2.42)

which is the same as

$$
\theta = \theta_{xy} = \frac{\pi}{2} - \arcsin \frac{2E_{XY}}{\sqrt{1 + 2E_{XX} \sqrt{1 + 2E_{YY}}}}.
$$

(2.43)

The increment of the final angle with respect to its initial value results in

$$
\Delta \Theta_{XY} = \theta_{xy} - \Theta_{XY} = -\arcsin \frac{2E_{XY}}{\sqrt{1 + 2E_{XX} \sqrt{1 + 2E_{YY}}}}.
$$

(2.44)

Analogous results are obtained starting from pairs of segments that are oriented in different combinations of the coordinate axes, resulting in...
\[
\begin{align*}
\Delta \Theta_{XY} &= - \arcsin \frac{2E_{XY}}{\sqrt{1 + 2E_{XX}} \sqrt{1 + 2E_{YY}}} \\
\Delta \Theta_{XZ} &= - \arcsin \frac{2E_{XZ}}{\sqrt{1 + 2E_{XX}} \sqrt{1 + 2E_{ZZ}}} \\
\Delta \Theta_{YZ} &= - \arcsin \frac{2E_{YZ}}{\sqrt{1 + 2E_{YY}} \sqrt{1 + 2E_{ZZ}}}
\end{align*}
\] (2.45)

Remark 2.10. The components \(E_{XY}\), \(E_{XZ}\) and \(E_{YZ}\) (or \(E_{12}\), \(E_{13}\) and \(E_{23}\)) of the tensor \(E\) (denoted angular strains) contain the information on variation of the angles between the differential segments that were initially (in the reference configuration) oriented in the directions \(X\), \(Y\) and \(Z\), respectively.

- If \(E_{XY} = 0\): The deformation does not produce a variation in the angle between the two segments initially oriented in the directions \(X\) and \(Y\).
- If \(E_{XZ} = 0\): The deformation does not produce a variation in the angle between the two segments initially oriented in the directions \(X\) and \(Z\).
- If \(E_{YZ} = 0\): The deformation does not produce a variation in the angle between the two segments initially oriented in the directions \(Y\) and \(Z\).

The physical interpretation of the components of the material strain tensor is shown in Figure 2.8 on an elemental parallelepiped in the neighborhood of a particle \(P\) with edges oriented in the direction of the coordinate axes.

2.7.2 Spatial Strain Tensor

Arguments similar to those of the previous subsection allow interpreting the spatial components of the strain tensor,

\[
e \equiv \begin{bmatrix}
e_{xx} & e_{xy} & e_{xz} \\
e_{xy} & e_{yy} & e_{yz} \\
e_{xz} & e_{yz} & e_{zz}
\end{bmatrix} = \begin{bmatrix}
e_{11} & e_{12} & e_{13} \\
e_{12} & e_{22} & e_{23} \\
e_{13} & e_{23} & e_{33}
\end{bmatrix}
\] (2.46)

The components of the main diagonal (longitudinal strains) can be interpreted in terms of the stretches and unit elongations of the differential segments ori-
Physical Interpretation of the Strain Tensors

Figure 2.8: Physical interpretation of the material strain tensor.

\[
\begin{align*}
\lambda_1 &= \frac{1}{\sqrt{1 - 2e_{11}}} = \frac{1}{\sqrt{1 - 2e_{xx}}} \Rightarrow \varepsilon_x = \frac{1}{\sqrt{1 - 2e_{xx}}} - 1 \\
\lambda_2 &= \frac{1}{\sqrt{1 - 2e_{22}}} = \frac{1}{\sqrt{1 - 2e_{yy}}} \Rightarrow \varepsilon_y = \frac{1}{\sqrt{1 - 2e_{yy}}} - 1 \\
\lambda_3 &= \frac{1}{\sqrt{1 - 2e_{33}}} \Rightarrow \varepsilon_z = \frac{1}{\sqrt{1 - 2e_{zz}}} - 1
\end{align*}
\]

(2.47)

while the components outside the main diagonal (angular strains) contain information on the variation of the angles between the differential segments oriented in the direction of the coordinate axes in the present configuration,

\[
\begin{align*}
\Delta \theta_{xy} &= \frac{\pi}{2} - \Theta_{XY} = -\arcsin \frac{2e_{xy}}{\sqrt{1 - 2e_{xx}} \sqrt{1 - 2e_{yy}}} \\
\Delta \theta_{xz} &= \frac{\pi}{2} - \Theta_{XZ} = -\arcsin \frac{2e_{xz}}{\sqrt{1 - 2e_{xx}} \sqrt{1 - 2e_{zz}}} \\
\Delta \theta_{yz} &= \frac{\pi}{2} - \Theta_{YZ} = -\arcsin \frac{2e_{yz}}{\sqrt{1 - 2e_{yy}} \sqrt{1 - 2e_{zz}}}
\end{align*}
\]

(2.48)

Figure 2.9 summarizes the physical interpretation of the components of the spatial strain tensor.
2.8 Polar Decomposition

The polar decomposition theorem of tensor analysis establishes that, given a second-order tensor $F$ such that $|F| > 0$, there exist an orthogonal tensor $Q$ and two symmetric tensors $U$ and $V$ such that

$$
\begin{align*}
U &= \sqrt{F^T \cdot F} \\
V &= \sqrt{F \cdot F^T} \\
Q &= F \cdot U^{-1} = V^{-1} \cdot F
\end{align*}
$$

This decomposition is unique for each tensor $F$ and is denominated left polar decomposition ($F = Q \cdot U$) or right polar decomposition ($F = V \cdot Q$). Tensors $U$ and $V$ are named right and left stretch tensors, respectively.

Considering now the deformation gradient tensor and the fundamental relation $d\mathbf{x} = F \cdot d\mathbf{X}$ defined in (2.2) as well as the polar decomposition given in (2.49), the following is obtained:

$$
d\mathbf{x} = F \cdot d\mathbf{X} = (V \cdot Q) \cdot d\mathbf{X} = V \cdot (Q \cdot d\mathbf{X})
$$

$$
\mathbf{F} (\bullet) \equiv \text{stretching} \circ \text{rotation} (\bullet)
$$

---

7 A second-order tensor $Q$ is orthogonal if $Q^T \cdot Q = Q \cdot Q^T = 1$ is verified.

8 To obtain the square root of a tensor, first the tensor must be diagonalized, then the square root of the elements in the diagonal of the diagonalized component matrix are obtained and, finally, the diagonalization is undone.

9 The notation $(\circ)$ is used here to indicate the composition of two operations $\xi$ and $\varphi$: $z = \varphi \circ \xi (x)$. 
Polar Decomposition

\[ d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = (\mathbf{Q} \cdot \mathbf{U}) \cdot d\mathbf{X} = \mathbf{Q} \cdot (\mathbf{U} \cdot d\mathbf{X}) \] (2.51)

**Remark 2.11.** An orthogonal tensor \( \mathbf{Q} \) (such that \( |\mathbf{Q}| = 1 \)) is named rotation tensor and the mapping \( \mathbf{y} = \mathbf{Q} \cdot \mathbf{x} \) is denominated rotation. A rotation has the following properties:

- When applied on any vector \( \mathbf{x} \), the result is another vector \( \mathbf{y} = \mathbf{Q} \cdot \mathbf{x} \) with the same modulus,
  \[ \|\mathbf{y}\|^2 = \mathbf{y} \cdot \mathbf{y} = [\mathbf{y}]^T \cdot [\mathbf{y}] = [\mathbf{Q} \cdot \mathbf{x}]^T \cdot [\mathbf{Q} \cdot \mathbf{x}] \equiv \mathbf{x} \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2. \]

- The result of multiplying (mapping) the orthogonal tensor \( \mathbf{Q} \) to two vectors \( \mathbf{x}^{(1)} \) and \( \mathbf{x}^{(2)} \) with the same origin and that form an angle \( \alpha \) between them, maintains the same angle between the images \( \mathbf{y}^{(1)} = \mathbf{Q} \cdot \mathbf{x}^{(1)} \) and \( \mathbf{y}^{(2)} = \mathbf{Q} \cdot \mathbf{x}^{(2)} \),
  \[ \frac{\mathbf{y}^{(1)} \cdot \mathbf{y}^{(2)}}{\|\mathbf{y}^{(1)}\| \cdot \|\mathbf{y}^{(2)}\|} = \frac{\mathbf{x}^{(1)} \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \mathbf{x}^{(2)}}{\|\mathbf{x}^{(1)}\| \cdot \|\mathbf{x}^{(2)}\|} = \frac{\mathbf{x}^{(1)} \cdot \mathbf{x}^{(2)}}{\|\mathbf{x}^{(1)}\| \cdot \|\mathbf{x}^{(2)}\|} = \cos \alpha. \]

In consequence, the mapping (rotation) \( \mathbf{y} = \mathbf{Q} \cdot \mathbf{x} \) maintains the angles and distances.

**Remark 2.12.** Equations (2.50) establish that the relative motion in the neighborhood of the particle during the deformation process (characterized by tensor \( \mathbf{F} \)) can be understood as the composition of a rotation (characterized by the rotation tensor \( \mathbf{Q} \), which maintains angles and distances) and a stretching or deformation in itself (which modifies angles and distances) characterized by the tensor \( \mathbf{V} \) (see Figure 2.10).
2.13 Remark. Alternatively, equations (2.51) allow characterizing the relative motion in the neighborhood of a particle during the deformation process as the superposition of a stretching or deformation in itself (characterized by tensor $\mathbf{U}$) and a rotation (characterized by the rotation tensor $\mathbf{Q}$).

A rigid body motion is a particular case of deformation characterized by $\mathbf{U} = \mathbf{V} = 1$ and $\mathbf{Q} = \mathbf{F}$.

2.9 Volume Variation

Consider a particle $P$ of the continuous medium in the reference configuration ($t = 0$) which has a differential volume $dV_0$ associated with it (see Figure 2.11). This differential volume is characterized by the positions of another three particles $Q$, $R$ and $S$ belonging to the differential neighborhood of $P$, which are aligned with this particle in three arbitrary directions. The volume differential $dV_t$, associated with the same particle in the present configuration (at time $t$), will also be characterized by the spatial points $P'$, $Q'$, $R'$ and $S'$ corresponding to Figure 2.11 (the positions of which define a parallelepiped that is no longer oriented along the coordinate axes).

The relative position vectors between the particles in the material configuration are $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ and $d\mathbf{X}^{(3)}$, and their counterparts in the spatial configuration...
tion are \(dx^{(1)} = F \cdot dX^{(1)}, dx^{(2)} = F \cdot dX^{(2)}\) and \(dx^{(3)} = F \cdot dX^{(3)}\). Obviously, the relations
\[
\begin{align*}
&dx^{(i)} = F \cdot dX^{(i)} \\
&dx^j = F_{jk} dX^k, \quad i, j, k \in \{1, 2, 3\}
\end{align*}
\] (2.52)
are satisfied. Then, the volumes\(^{10}\) associated with a particle in both configurations can be written as
\[
\begin{align*}
dV_0 &= \left(dX^{(1)} \times dX^{(2)}\right) \cdot dX^{(3)} = \det \begin{bmatrix} dX_1^{(1)} & dX_2^{(1)} & dX_3^{(1)} \\ dX_1^{(2)} & dX_2^{(2)} & dX_3^{(2)} \\ dX_1^{(3)} & dX_2^{(3)} & dX_3^{(3)} \end{bmatrix} = \det[M], \\
dV_t &= \left(dx^{(1)} \times dx^{(2)}\right) \cdot dx^{(3)} = \det \begin{bmatrix} dx_1^{(1)} & dx_2^{(1)} & dx_3^{(1)} \\ dx_1^{(2)} & dx_2^{(2)} & dx_3^{(2)} \\ dx_1^{(3)} & dx_2^{(3)} & dx_3^{(3)} \end{bmatrix} = \det[m],
\end{align*}
\] (2.53)
where \(M_{ij} = dx_j^{(i)}\) and \(m_{ij} = dx_j\). Considering these expressions,
\[
m_{ij} = dx_j^{(i)} = F_{jk} dX^k = dM_{ik} F_{kj} = dM_{ik} F^T_{kj} \implies m = M \cdot F^T \tag{2.54}
\]
is deduced and, consequently\(^{11}\),
\[
\begin{align*}
&dV_t = \left|m\right| = \left| M \cdot F^T \right| = \left| M \right| \left| F^T \right| = \left| F \right| \left| M \right| = \left| F \right| \left| dV_0 \right| \\
&dV_t = dV(x(X,t),t) = \left| F(x,t) \right| dV_0(x,0) = \left| F \right| dV_0 \\
&\implies dV_t = \left| F \right| dV_0 \tag{2.55}
\end{align*}
\]
\(^{10}\) The volume of a parallelepiped is calculated as the scalar triple product \((a \times b) \cdot c\) of the concurrent edge-vectors \(a, b\) and \(c\), which meet at any of the parallelepiped’s vertices. Note that the scalar triple product is the determinant of the matrix constituted by the components of the above mentioned vectors arranged in rows.
\(^{11}\) The expressions \(\left| A \cdot B \right| = \left| A \right| \cdot \left| B \right|\) and \(\left| A^T \right| = \left| A \right|\) are used here.
2.10 Area Variation

Consider an area differential \( dA \) associated with a particle \( P \) in the reference configuration and its variation along time. To define this area differential, consider two particles \( Q \) and \( R \) in the differential neighborhood of \( P \), whose relative positions with respect to this particle are \( dX^{(1)} \) and \( dX^{(2)} \), respectively (see Figure 2.12). Consider also an arbitrary auxiliary particle \( S \) whose relative position vector is \( dX^{(3)} \). An area differential vector \( dA = dA \mathbf{N} \) associated with the scalar differential area, \( dA \), is defined. The module of vector \( dA \) is \( dA \) and its direction is the same as that of the unit normal vector in the material configuration \( \mathbf{N} \).

In the present configuration, at time \( t \), the particle will occupy a point in space \( P' \) and will have an area differential \( da \) associated with it which, in turn, defines an area differential vector \( da = da \mathbf{n} \), where \( \mathbf{n} \) is the corresponding unit normal vector in the spatial configuration. Consider also the positions of the other particles \( Q', R' \) and \( S' \) and their relative position vectors \( dx^{(1)} \), \( dx^{(2)} \) and \( dx^{(3)} \).

The volumes \( dV_0 \) and \( dV_t \) of the corresponding parallelepipeds can be calculated as

\[
\begin{align*}
  dV_0 &= dH dA = \overbrace{dX^{(3)} \cdot \mathbf{N}}^{dH} dA = \overbrace{dX^{(3)} \cdot \mathbf{N} dA}^{dA} = \overbrace{dA \cdot dX^{(3)}}^{da} \\
  dV_t &= dh da = \overbrace{dx^{(3)} \cdot \mathbf{n}}^{dh} da = \overbrace{dx^{(3)} \cdot \mathbf{n} da}^{da} = da \cdot dx^{(3)}
\end{align*}
\]

and, taking into account that \( dx^{(3)} = \mathbf{F} \cdot dX^{(3)} \), as well as the expression for change in volume (2.55), results in

\[
da \cdot \mathbf{F} \cdot dX^{(3)} = da \cdot dx^{(3)} = dV_t = |\mathbf{F}| dV_0 = |\mathbf{F}| dA \cdot dX^{(3)} \quad \forall dX^{(3)} .
\]
Comparing the first and last terms in (2.57) and considering that the relative position of particle $S$ can take any value (as can, therefore, vector $d\mathbf{X}^{(3)}$), finally yields

$$da \cdot \mathbf{F} = |\mathbf{F}| d\mathbf{A} \implies da = |\mathbf{F}| d\mathbf{A} \cdot \mathbf{F}^{-1}$$

(2.58)

To obtain the relation between the two area differential scalars, $d\mathbf{A}$ and $da$, expressions $d\mathbf{A} = N d\mathbf{A}$ and $da = n da$ are replaced into (2.58) and the modules are taken, resulting in

$$da \cdot n = |\mathbf{F}| N \cdot d\mathbf{F}^{-1} d\mathbf{A} \implies da = |\mathbf{F}| \|N \cdot d\mathbf{F}^{-1}\| d\mathbf{A}.$$  

(2.59)

### 2.11 Infinitesimal Strain

Infinitesimal strain theory (also denominated *small deformation theory*) is based on two simplifying hypotheses of the general theory (or *finite strain theory*) seen in the previous sections (see Figure 2.13).

**Definition 2.5.** The simplifying hypotheses are:

1) *Displacements are very small* compared to the typical dimensions in the continuous medium: $\|\mathbf{u}\| < < \|\mathbf{X}\|$.  

2) *Displacement gradients are very small* (infinitesimal).
In accordance with the first hypothesis, the reference configuration $\Omega_0$ and the present configuration $\Omega_t$ are very close together and are considered to be indistinguishable from one another. Consequently, the material and spatial coordinates coincide and discriminating between material and spatial descriptions no longer makes sense.

\[
\begin{align*}
\mathbf{x} &= \mathbf{X} + \mathbf{u} 
\Rightarrow
\mathbf{X}_i &= \mathbf{X}_i + u_i 
\Rightarrow
\mathbf{u}_i &= \mathbf{U}_i 
\Rightarrow
\frac{\partial \mathbf{u}_i}{\partial \mathbf{X}_j} &= \frac{\partial \mathbf{U}_i}{\partial \mathbf{X}_j} 
\Rightarrow
\mathbf{J}_{ij} &= \mathbf{J} (2.62)
\end{align*}
\]

The second hypothesis can be written in mathematical form as

\[
\left| \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} \right| \ll 1 \quad \forall i, j \in \{1, 2, 3\} .
\]

(2.61)

### 2.11.1 Strain Tensors. Infinitesimal Strain Tensor

The material and spatial displacement gradient tensors coincide. Indeed, in view of (2.60),

\[
\begin{align*}
x_j &= X_j \\
u_i (\mathbf{x}, t) &= U_i (\mathbf{X}, t) 
\Rightarrow
j_{ij} &= \frac{\partial u_i}{\partial x_j} = \frac{\partial U_i}{\partial X_j} = J_{ij} \quad \Rightarrow \quad \mathbf{j} = \mathbf{J} (2.62)
\end{align*}
\]

and the material strain tensor results in
Infinitesimal Strain

\[
\begin{align*}
E &= \frac{1}{2} \left( J + J^T + J \cdot J \right) \approx \frac{1}{2} \left( J + J^T \right), \\
E_{ij} &= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \approx \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \\
\end{align*}
\]

(2.63)

where the infinitesimal character of the second-order term \( \partial u_i \partial u_k / \partial x_j \partial x_i \) has been taken into account. Operating in a similar manner with the spatial strain tensor,

\[
\begin{align*}
e &= \frac{1}{2} \left( j + j^T - j^T \cdot j \right) \approx \frac{1}{2} \left( J + J^T \right), \\
e_{ij} &= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \approx \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \\
\end{align*}
\]

(2.64)

Equations (2.63) and (2.64) allow defining the infinitesimal strain tensor (or small strain tensor) \( \mathbf{\varepsilon} \) as

\[
\mathbf{\varepsilon} = \frac{1}{2} \left( \mathbf{J} + \mathbf{J}^T \right) = \frac{1}{2} \left( \mathbf{J} + \mathbf{J}^T \right) \text{not} \quad \mathbf{\varepsilon}^T = \frac{1}{2} \left( \mathbf{J} + \mathbf{J}^T \right)^T = \frac{1}{2} \left( \mathbf{J}^T + \mathbf{J} \right) = \mathbf{\varepsilon}
\]

(2.65)

**Remark 2.14.** Under the infinitesimal strain hypothesis, the material and spatial strain tensors coincide and collapse into the infinitesimal strain tensor.

\[ \mathbf{E}(\mathbf{x}, t) = \mathbf{e}(\mathbf{x}, t) = \mathbf{\varepsilon}(\mathbf{x}, t) \]

**Remark 2.15.** The infinitesimal strain tensor is symmetric, as observed in its definition in (2.65).

\[ \mathbf{\varepsilon}^T = \frac{1}{2} \left( \mathbf{J} + \mathbf{J}^T \right)^T = \frac{1}{2} \left( \mathbf{J}^T + \mathbf{J} \right) = \mathbf{\varepsilon} \]

13 The symmetric gradient operator \( \nabla^s \) is defined as \( \nabla^s (\bullet) = (\langle \bullet \rangle \otimes \nabla + \nabla \otimes (\bullet)) / 2. \)
Remark 2.16. The components of the infinitesimal strain tensor $\varepsilon$ are infinitesimal ($\varepsilon_{ij} \ll 1$). Proof is obvious from (2.65) and the condition that the components of $J = j$ are infinitesimal (see (2.61)).

Example 2.4 – Determine under which conditions the motion in Example 2.1 constitutes an infinitesimal strain case and obtain the infinitesimal strain tensor for this case. Compare it with the result obtained from the spatial and material strain tensors in Example 2.2 taking into account the infinitesimal strain hypotheses.

Solution

The equation of motion is given by

\[
\begin{aligned}
x_1 &= X_1 - Ax_3 \\
x_2 &= X_2 - Ax_3 \\
x_3 &= -Ax_1 + Ax_2 + x_3
\end{aligned}
\]

from which the displacement field is obtained,

\[
U(X,t) = x - X \equiv \begin{bmatrix} U_1 = -Ax_3 \\ U_2 = -Ax_3 \\ U_3 = -Ax_1 + Ax_2 \end{bmatrix}.
\]

It is obvious that, for the displacements to be infinitesimal, $A$ must be infinitesimal ($A \ll 1$). Now, to obtain the infinitesimal strain tensor, first the displacement gradient tensor $J(X,t) = j(x,t)$ must be computed,

\[
J = U \otimes \nabla \equiv \begin{bmatrix} -Ax_3 \\ -Ax_3 \\ -Ax_1 + Ax_2 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -A \\ 0 & 0 & -A \\ -A & A & 0 \end{bmatrix}.
\]

Then, the infinitesimal strain tensor, in accordance to (2.65), is

\[
\varepsilon = \nabla U \equiv \begin{bmatrix} 0 & 0 & -A \\ 0 & 0 & 0 \\ -A & 0 & 0 \end{bmatrix}.
\]
The material and spatial strain tensors obtained in Example 2.2 are, respectively,
\[
E(X,t) \equiv \frac{1}{2} \begin{bmatrix}
A^2 & -A^2 & -2A \\
-A^2 & A^2 & 0 \\
-2A & 0 & 2A^2 \\
\end{bmatrix}
\]
and
\[
e(X,t) \equiv \frac{1}{2} \begin{bmatrix}
-3A^2 - 2A^4 & A^2 + 2A^4 & -2A - 2A^3 \\
A^2 + 2A^4 & A^2 - 2A^4 & 2A^3 \\
-2A - 2A^3 & 2A^3 & -2A^2 \\
\end{bmatrix}.
\]
Neglecting the second-order and higher-order infinitesimal terms \(A^4 \ll A^3 \ll A^2 \ll A\) results in
\[
E \equiv \begin{bmatrix}
0 & 0 & -A \\
0 & 0 & -A \\
-A & A & 0 \\
\end{bmatrix}
\]
and
\[
e \equiv \begin{bmatrix}
0 & 0 & -A \\
0 & 0 & -A \\
-A & A & 0 \\
\end{bmatrix} \Rightarrow E = e - \varepsilon,
\]
which is in accordance with Remark 2.14.

### 2.11.2 Stretch. Unit Elongation

Considering the general expression (2.30) of the unit elongation in the direction \(T \equiv t\) \((\lambda_t = \sqrt{1 + 2\mathbf{t} \cdot \mathbf{E} \cdot \mathbf{t}})\) and applying a Taylor series expansion\(^{14}\) around 0 (taking into account that \(E = \varepsilon\) is infinitesimal and, therefore, so is \(x = \mathbf{t} \cdot \varepsilon \cdot \mathbf{t} \)), yields
\[
\lambda_t = \sqrt{1 + 2\mathbf{t} \cdot \varepsilon \cdot \mathbf{t}} \cong 1 + \mathbf{t} \cdot \varepsilon \cdot \mathbf{t} \tag{2.66}
\]
\[
\epsilon_{11} = \lambda_t - 1 = \mathbf{t} \cdot \varepsilon \cdot \mathbf{t}
\]

### 2.11.3 Physical Interpretation of the Infinitesimal Strains

Consider the infinitesimal strain tensor \(\varepsilon\) and its components in the coordinate system \(x_1 \equiv x, x_2 \equiv y, x_3 \equiv z\), shown in Figure 2.14,
\[
\varepsilon \equiv \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \\
\end{bmatrix} = \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \\
\end{bmatrix}. \tag{2.67}
\]

\(^{14}\) The Taylor series expansion of \(\sqrt{1+x}\) around \(x = 0\) is \(\sqrt{1+x} = 1 + x/2 + O(x^2)\).
Consider a differential segment $PQ$ oriented in the reference configuration parallel to the coordinate axis $x_1 \equiv x$. The stretch $\lambda_x$ and the unit elongation $\varepsilon_x$ in this direction are, according to (2.66) with $t = [1, 0, 0]^T$,

$$\lambda_x = 1 + t \cdot \varepsilon \cdot t = 1 + \varepsilon_{xx} \implies \varepsilon_x = \lambda_x - 1 = \varepsilon_{xx}.$$

(2.68)

This allows assigning to the component $\varepsilon_{xx} \equiv \varepsilon_{11}$ the physical meaning of unit elongation $\varepsilon_x$ in the direction of the coordinate axis $x_1 \equiv x$. A similar interpretation is deduced for the other components in the main diagonal of the tensor $\varepsilon_x (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz})$,

$$\varepsilon_{xx} = \varepsilon_x; \quad \varepsilon_{yy} = \varepsilon_y; \quad \varepsilon_{zz} = \varepsilon_z.$$

(2.69)

Given now the components outside the main diagonal of $\varepsilon$, consider the differential segments $PQ$ and $PR$ oriented in the reference configuration parallel to the coordinate directions $x$ and $y$, respectively. Then, these two segments form an angle $\Theta_{xy} = \pi/2$ in this configuration. Applying (2.43), the increment in the corresponding angle results in

$$\Delta \theta_{xy} = \theta_{xy} - \pi/2 = -2 \arcsin \frac{\varepsilon_{xy}}{\sqrt{1 + 2\varepsilon_{xx}}} \approx -2 \varepsilon_{xy},$$

(2.70)

where the infinitesimal character of $\varepsilon_{xx}$, $\varepsilon_{yy}$ and $\varepsilon_{xy}$ has been taken into account. Consequently, $\varepsilon_{xy}$ can be interpreted from (2.70) as minus the semi-increment, produced by the strain, of the angle between the two differential segments initially oriented parallel to the coordinate directions $x$ and $y$. A similar interpretation is deduced for the other components $\varepsilon_{xz}$ and $\varepsilon_{yz}$,

$$\varepsilon_{xy} = -\frac{1}{2} \Delta \theta_{xy} \quad ; \quad \varepsilon_{xz} = -\frac{1}{2} \Delta \theta_{xz} \quad ; \quad \varepsilon_{yz} = -\frac{1}{2} \Delta \theta_{yz}.$$

(2.71)

---

15 The Taylor series expansion of $\arcsin x$ around $x = 0$ is $\arcsin x = x + O(x^3)$.  

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2.11.4 Engineering Strains. Vector of Engineering Strains

There is a strong tradition in engineering to use a particular denomination for the components of the infinitesimal strain tensor. This convention is named engineering notation, as opposed to the scientific notation generally used in continuum mechanics. Both notations are synthesized as follows.

\[
\varepsilon \equiv \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}
\end{bmatrix} \equiv \begin{bmatrix}
\varepsilon_{xx} & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\
\frac{1}{2} \gamma_{xy} & \varepsilon_{yy} & \frac{1}{2} \gamma_{yz} \\
\frac{1}{2} \gamma_{xz} & \frac{1}{2} \gamma_{yz} & \varepsilon_{zz}
\end{bmatrix}
\]

Remark 2.17. The components in the main diagonal of the strain tensor (named longitudinal strains) are denoted by \(\varepsilon(\cdot)\) and coincide with the unit elongations in the directions of the coordinate axes. Positive values of longitudinal strains (\(\varepsilon(\cdot) > 0\)) correspond to an increase in length of the corresponding differential segments in the reference configuration.

Remark 2.18. The components outside the main diagonal of the strain tensor are characterized by the values \(\gamma(\cdot, \cdot)\) (named angular strains) and can be interpreted as the decrements of the corresponding angles oriented in the Cartesian directions of the reference configuration. Positive values of angular strains (\(\gamma(\cdot, \cdot) > 0\)) indicate that the corresponding angles close with the deformation process.

In engineering, it is also frequent to exploit the symmetry of the infinitesimal strain tensor (see Remark 2.15) to work only with the six components of the tensor that are different, grouping them in the vector of engineering strains, which is defined as follows.

\[
\varepsilon \in \mathbb{R}^6 \quad \varepsilon \overset{def}{=} \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} & \frac{1}{2} \gamma_{yz}
\end{bmatrix}^T
\]
2.11.5 Variation of the Angle between Two Differential Segments in Infinitesimal Strain

Consider any two differential segments, \( \overline{PQ} \) and \( \overline{PR} \), in the reference configuration and the angle \( \Theta \) they define (see Figure 2.15). The angle formed by the corresponding deformed segments in the present configuration is \( \theta = \Theta + \Delta \theta \). Applying (2.42) to this case results in

\[
\cos \theta = \cos (\Theta + \Delta \theta) = \frac{T^{(1)} \cdot (1 + 2\mathbf{e}) \cdot T^{(2)}}{\sqrt{1 + 2T^{(1)} \cdot \mathbf{e} \cdot T^{(1)}} \sqrt{1 + 2T^{(2)} \cdot \mathbf{e} \cdot T^{(2)}}},
\]

where \( T^{(1)} \) and \( T^{(2)} \) are the unit vectors in the directions of \( \overline{PQ} \) and \( \overline{PR} \) and, therefore, the relation \( T^{(1)} \cdot T^{(2)} = \left\| T^{(1)} \right\| \left\| T^{(2)} \right\| \cos \Theta = \cos \Theta \) is fulfilled. Considering the infinitesimal character of the components of \( \mathbf{e} \) and \( \Delta \theta \), the following holds true

\[
\cos \theta = \cos (\Theta + \Delta \theta) = \cos \Theta \cdot \cos \Delta \theta - \sin \Theta \cdot \sin \Delta \theta \approx \cos \Theta - \sin \Theta \cdot \Delta \theta
\]

\[
= \cos \Theta - \sin \Theta \cdot \Delta \theta = \frac{T^{(1)} \cdot T^{(2)} + 2T^{(1)} \cdot \mathbf{e} \cdot T^{(2)}}{\sqrt{1 + T^{(1)} \cdot \mathbf{e} \cdot T^{(1)}} \sqrt{1 + T^{(2)} \cdot \mathbf{e} \cdot T^{(2)}}}
\]

\[
\approx 1 \approx 1
\]

\[
\Delta \theta \approx -2T^{(1)} \cdot \mathbf{e} \cdot T^{(2)}
\]

Therefore, \( \sin \Theta \cdot \Delta \theta = -2T^{(1)} \cdot \mathbf{e} \cdot T^{(2)} \) and

\[
\frac{\Delta \theta}{\sin \Theta} = -\frac{2T^{(1)} \cdot \mathbf{e} \cdot T^{(2)}}{\sin \Theta} = -\frac{2t^{(1)} \cdot \mathbf{e} \cdot t^{(2)}}{\sin \theta},
\]

where the infinitesimal character of the strain has been taken into account and, thus, it follows that \( T^{(1)} \approx t^{(1)} \), \( T^{(2)} \approx t^{(2)} \) and \( \Theta \approx \theta \).

2.11.6 Polar Decomposition

The polar decomposition of the deformation gradient tensor \( F \) is given by (2.49) for the general case of finite strain. In the case of infinitesimal strain, recall-

\[X. Oliver and C. Agelet de Saracibar Continuum Mechanics for Engineers. Theory and Problems doi:10.13140/RG.2.2.25821.20961]"
Figure 2.15: Variation of the angle between two differential segments in infinitesimal strain.

In a similar manner, due to the infinitesimal character of the components of the tensor \( \varepsilon \) (see Remark 2.16), the inverse of tensor \( U \) results in

\[
U^{-1} = (1 + \varepsilon)^{-1} = 1 - \varepsilon = 1 - \frac{1}{2} (J + J^T). 
\]  
(2.78)

Therefore, the rotation tensor \( Q \) in (2.49) can be written as

\[
Q = F : U^{-1} = (1 + \varepsilon)^{-1} \left( 1 - \frac{1}{2} (J + J^T) \right) = 
\]

\[
1 + J - \frac{1}{2} (J + J^T) - \frac{1}{2} J : (J + J^T) = 1 + \frac{1}{2} (J - J^T) 
\]  

\[
\Rightarrow Q = 1 + \Omega. 
\]  
(2.79)

---

17 The Taylor series expansions of tensor \( \sqrt{1+x} \) around \( x = 0 \) is \( \sqrt{1+x} = 1 + x/2 + O(x^2) \).

18 The Taylor series expansions of tensor \( (1+x)^{-1} \) around \( x = 0 \) is \( (1+x)^{-1} = 1 - x + O(x^3) \).
Equation (2.79) defines the *infinitesimal rotation tensor* $\Omega$ as follows:

$$\Omega \overset{\text{def}}{=} \frac{1}{2} (J - J^T) = \frac{1}{2} (u \otimes \nabla - \nabla \otimes u) \overset{\text{def}}{=} \nabla^a u$$

$$\Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \ll 1 \quad i, j \in \{1, 2, 3\}$$

(2.80)

**Remark 2.19.** The tensor $\Omega$ is antisymmetric. Indeed,

$$\begin{align*}
\Omega^T &= \frac{1}{2} (J - J^T)^T = \frac{1}{2} (J^T - J) = -\Omega \\
\Omega_{ji} &= -\Omega_{ij} \quad i, j \in \{1, 2, 3\}
\end{align*}$$

Consequently, the terms in the main diagonal of $\Omega$ are zero, and its matrix of components has the structure

$$[\Omega] = \begin{bmatrix}
0 & \Omega_{23} & -\Omega_{31} \\
-\Omega_{23} & 0 & \Omega_{12} \\
\Omega_{31} & -\Omega_{12} & 0
\end{bmatrix}$$

In a small rotation context, tensor $\Omega$ characterizes the rotation ($Q = I + \Omega$) and, thus, the denomination of infinitesimal rotation tensor. Since it is an antisymmetric tensor, it is defined solely by three different components ($\Omega_{23}, \Omega_{31}, \Omega_{12}$), which form the *infinitesimal rotation vector* $\theta$.

$$\theta = \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{bmatrix} = \begin{bmatrix}
-\Omega_{23} \\
-\Omega_{31} \\
-\Omega_{12}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\
\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\
\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}
\end{bmatrix} \overset{\text{def}}{=} \frac{1}{2} \nabla \times \mathbf{u} \quad (2.81)$$

---

19 The *antisymmetric gradient* operator $\nabla^a$ is defined as $\nabla^a (\bullet) = ([\bullet] \otimes \nabla - \nabla \otimes [\bullet]) / 2$.

20 The operator *rotational of* ($\bullet$) is denoted as $\nabla \times (\bullet)$.
Expressions (2.12), (2.65) and (2.79) allow writing
\[
F = 1 + J + \frac{1}{2} (J + J^T) + \frac{1}{2} (J - J^T) \implies F = 1 + \varepsilon + \Omega.
\] (2.82)

**Remark 2.20.** The results of performing a dot product of the infinitesimal rotation tensor \( \Omega \) and performing a cross product of the infinitesimal rotation vector \( \theta \) with any vector \( r \equiv [r_1, r_2, r_3]^T \) (see Figure 2.16) coincide. Indeed,

\[
\Omega \cdot r = \theta \times r
\]

Consequently, vector \( \Omega \cdot r = \theta \times r \) has the following characteristics:

- It is orthogonal to vector \( r \) (because it is the result of a vector product in which \( r \) is involved).
- Its module is infinitesimal (because \( \theta \) is infinitesimal).
- Vector \( r + \Omega \cdot r = r + \theta \times r \) can be considered, except for higher-order infinitesimals, as the result of applying a rotation \( \theta \) on vector \( r \).
Consider now a differential segment $d\mathbf{X}$ in the neighborhood of a particle $P$ in the reference configuration (see Figure 2.17). In accordance with (2.82), the stretching transforms this vector into vector $d\mathbf{x}$ as follows.

\[
d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = (1 + \varepsilon + \Omega) \cdot d\mathbf{X} = \varepsilon \cdot d\mathbf{X} + (1 + \Omega) \cdot d\mathbf{X}
\]

(2.83)

**Remark 2.21.** Under infinitesimal strain hypotheses, the expression in (2.83) characterizes the relative motion of a particle, in the differential neighborhood of this particle, as the following sum:

a) A stretching or deformation in itself, characterized by the infinitesimal strain tensor $\varepsilon$.

b) A rotation characterized by the infinitesimal rotation tensor $\Omega$ which, in the infinitesimal strain context, maintains angles and distances.

The superposition (stretching \(\circ\) rotation) of the general finite strain case (see Remark 2.12) degenerates, for the infinitesimal strain case, into a simple addition (stretching + rotation).
2.12 Volumetric Strain

**Definition 2.6.** The *volumetric strain* is the increment produced by the deformation of the volume associated with a particle, per unit of volume in the reference configuration.

This definition can be mathematically expressed as (see Figure 2.18)

\[
\text{Volumetric strain: } e(\mathbf{X}, t) \overset{\text{def}}{=} \frac{dV(\mathbf{X}, t) - dV(\mathbf{X}, 0)}{dV(\mathbf{X}, 0)} = \frac{dV_t - dV_0}{dV_0}. \tag{2.84}
\]

Figure 2.18: Volumetric strain.
Equation (2.55) allows expressing, in turn, the volumetric strain as follows:

- **Finite strain**

  \[ e = \frac{dV_t - dV_0}{dV_0} = \frac{|F|}{dV_0}dV_0 - dV_0 \quad \Rightarrow \quad e = |F| - 1 \quad (2.85) \]

- **Infinitesimal strain**

Considering (2.49) and recalling that \( Q \) is an orthogonal tensor \( (|Q| = 1) \), yields

\[ |F| = |Q \cdot U| = |Q||U| = |U| = |1 + \varepsilon| = \text{det} \begin{bmatrix} 1 + \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & 1 + \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & 1 + \varepsilon_{zz} \end{bmatrix}, \quad (2.86) \]

where (2.77) has been considered. Taking into account that the components of \( \varepsilon \) are infinitesimal, and neglecting in the expression of its determinant the second-order and higher-order infinitesimal terms, results in

\[ |F| = \text{det} \begin{bmatrix} 1 + \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & 1 + \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & 1 + \varepsilon_{zz} \end{bmatrix} = 1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} + O(e^2) \approx 1 + \text{Tr}(\varepsilon). \quad (2.87) \]

Then, introducing (2.87) into (2.85) yields, for the infinitesimal strain case

\[ \begin{aligned} dV_t &= (1 + \text{Tr}(\varepsilon))dV_0 \\
\frac{dV_t}{dV_0} &= \frac{|F|}{dV_0} - 1 \quad \Rightarrow \quad e = \text{Tr}(\varepsilon). \quad (2.88) \end{aligned} \]

### 2.13 Strain Rate

In the previous sections of this chapter, the concept of strain has been studied, defined as the variation of the relative position (angles and distances) of the particles in the neighborhood of a given particle. In the following sections, the rate at which this relative position changes will be considered by introducing the concept of strain rate as a measure of the variation in the relative position between particles per unit of time.
2.13.1 Velocity Gradient Tensor

Consider the configuration corresponding to a time $t$, two particles of the continuous medium $P$ and $Q$ that occupy the spatial points $P'$ and $Q'$ at said instant of time (see Figure 2.19), their velocities $v_P = v(x, t)$ and $v_Q = v(x + dx, t)$, and their relative velocity,

$$dv(x,t) = v_Q - v_P = v(x + dx, t) - v(x, t).$$  \hfill (2.89)

Then,

$$
\begin{align*}
    dv &= \frac{\partial v}{\partial x} \cdot dx = l \cdot dx, \\
    dv_i &= \frac{\partial v_i}{\partial x_j} dx_j = l_{ij} dx_j, \quad i, j \in \{1, 2, 3\}
\end{align*}
$$  \hfill (2.90)

where the spatial velocity gradient tensor $l(x, t)$ has been introduced.

Figure 2.19: Velocities of two particles in the continuous medium.
2.13.2 Strain Rate and Spin Tensors

The velocity gradient tensor can be split into a symmetric and an antisymmetric part,

\[ l = d + w, \]

where \( d \) is a symmetric tensor denominated strain rate tensor,

\[
\begin{align*}
    d & \overset{\text{def}}{=} \text{sym}(l) = \frac{1}{2} (l + l^T) = \frac{1}{2} (v \otimes \nabla + \nabla \otimes v) \Rightarrow \nabla^s v \\
    d_{ij} & = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad i, j \in \{1, 2, 3\} \\
    [d] & = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{bmatrix}
\end{align*}
\]

and \( w \) is an antisymmetric tensor denominated rotation rate tensor or spin tensor, whose expression is

\[
\begin{align*}
    w & \overset{\text{def}}{=} \text{skew}(l) = \frac{1}{2} (l - l^T) = \frac{1}{2} (v \otimes \nabla - \nabla \otimes v) \Rightarrow \nabla^a v \\
    w_{ij} & = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \quad i, j \in \{1, 2, 3\} \\
    [w] & = \begin{bmatrix} 0 & w_{12} & -w_{31} \\ -w_{12} & 0 & w_{23} \\ w_{31} & -w_{23} & 0 \end{bmatrix}
\end{align*}
\]

2.13.3 Physical Interpretation of the Strain Rate Tensor

Consider a differential segment defined by the particles \( P \) and \( Q \) of Figure 2.20 and the variation of their squared length along time,

\[
\frac{d}{dt} ds^2 = \frac{d}{dt} (dx \cdot dx) = \frac{d}{dt} (dx) \cdot dx + dx \cdot \frac{d}{dt} (dx) = d \left( \frac{dx}{dt} \right) \cdot dx + dx \cdot d \left( \frac{dx}{dt} \right) = dv \cdot dx + dx \cdot dv,
\]

21 Every second-order tensor \( a \) can be decomposed into the sum of its symmetric part \( (\text{sym}(a)) \) and its antisymmetric or skew-symmetric part \( (\text{skew}(a)) \) in the form:

\[ a = \text{sym}(a) + \text{skew}(a) \quad \text{with} \quad \text{sym}(a) = (a + a^T)/2 \quad \text{and} \quad \text{skew}(a) = (a - a^T)/2. \]
and using relations (2.90) and (2.93), the expression
\[
\frac{d}{dt} ds^2 = \left( \mathbf{d} \cdot \mathbf{d}^T \right) \cdot \mathbf{d} \mathbf{x} + \mathbf{d} \mathbf{x} \cdot ( \mathbf{l} \cdot \mathbf{d} \mathbf{x} ) = \mathbf{d} \mathbf{x} \cdot \left( \mathbf{l}^T + \mathbf{l} \right) \cdot \mathbf{d} \mathbf{x} = 2 \mathbf{d} \mathbf{x} \cdot \mathbf{d} \cdot \mathbf{d} \mathbf{x} \tag{2.96}
\]
is obtained. Differentiating now (2.20) with respect to time and taking into account (2.96) yields
\[
2 \mathbf{d} \mathbf{x} \cdot \mathbf{d} \cdot \mathbf{d} \mathbf{x} = \frac{d}{dt} \left( 2 \mathbf{d} \mathbf{x} \cdot \mathbf{E} (\mathbf{X}, t) \cdot \mathbf{d} \mathbf{x} \right) = 2 \mathbf{d} \mathbf{x} \cdot \frac{d\mathbf{E}}{dt} \cdot \mathbf{d} \mathbf{x} = 2 \mathbf{d} \mathbf{X} \cdot \mathbf{\dot{E}} \cdot \mathbf{d} \mathbf{X} \tag{2.97}
\]
Replacing (2.2) into (2.97) results in
\[
\mathbf{d} \mathbf{X} \cdot \mathbf{\dot{E}} \cdot \mathbf{d} \mathbf{X} = \mathbf{d} \mathbf{x} \cdot \mathbf{d} \cdot \mathbf{d} \mathbf{x} \equiv [\mathbf{d} \mathbf{x}]^T [\mathbf{d}] [\mathbf{d} \mathbf{x}] = [\mathbf{d} \mathbf{X}]^T \left[ \mathbf{F}^T \cdot \mathbf{d} \cdot \mathbf{F} \right] [\mathbf{d} \mathbf{X}]
\]
\[
\Rightarrow \quad \mathbf{d} \mathbf{X} \cdot \left( \mathbf{F}^T \cdot \mathbf{d} \cdot \mathbf{F} - \mathbf{\dot{E}} \right) \cdot \mathbf{d} \mathbf{X} = 0 \quad \forall \mathbf{d} \mathbf{X} \quad \Rightarrow \quad \mathbf{F}^T \cdot \mathbf{d} \cdot \mathbf{F} = \dot{\mathbf{E}} = 0 \quad \Rightarrow \quad \dot{\mathbf{E}} = \mathbf{F}^T \cdot \mathbf{d} \cdot \mathbf{F} \tag{2.98}
\]

**Remark 2.22.** Equation (2.98) shows the existing relationship between the strain rate tensor $\mathbf{d} (\mathbf{x}, t)$ and the material derivative of the material strain tensor $\mathbf{\dot{E}} (\mathbf{X}, t)$, providing a physical interpretation (and justifying the denomination) of tensor $\mathbf{d} (\mathbf{x}, t)$. However, the same equation reveals that tensors $\mathbf{d} (\mathbf{x}, t)$ and $\mathbf{\dot{E}} (\mathbf{X}, t)$ are not exactly the same. Both tensors will coincide in the following cases:

- In the reference configuration: $t = t_0 \Rightarrow \mathbf{F} \big|_{t = t_0} = \mathbf{1}$.
- In infinitesimal strain theory: $\mathbf{x} \approx \mathbf{X} \Rightarrow \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \approx \mathbf{1}$.

---

22 Here, the following tensor algebra theorem is used: given a second-order tensor $\mathbf{A}$, if $\mathbf{x} \cdot \mathbf{A} \cdot \mathbf{x} = 0$ is verified for all vectors $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{A} = \mathbf{0}$. 
2.13.4 Physical Interpretation of the Rotation Rate Tensor

Taking into account the antisymmetric character of \( \mathbf{w} \) (which implies it can be defined using only three different components), the vector

\[
\mathbf{\omega} = \frac{1}{2} \mathbf{\text{rot}} (\mathbf{v}) = \frac{1}{2} \nabla \times \mathbf{v} = \frac{1}{2} \begin{bmatrix}
\frac{\partial v_2}{\partial x_3} & -\frac{\partial v_3}{\partial x_2} & \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \\
\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3} & \frac{\partial v_1}{\partial x_3} & -\frac{\partial v_2}{\partial x_1} \\
\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_2} - \frac{\partial v_1}{\partial x_3}
\end{bmatrix} = \begin{bmatrix}
-w_{23} \\
-w_{31} \\
-w_{12}
\end{bmatrix}
\]

is extracted from (2.94). Vector \( 2 \mathbf{\omega} = \nabla \times \mathbf{v} \) is named vorticity vector \(^{23}\). It can be proven (in an analogous manner to Remark 2.20) that the equality

\[
\mathbf{\omega} \times \mathbf{r} = \mathbf{w} \cdot \mathbf{r} \quad \forall \mathbf{r}
\]

(2.100)
is satisfied. Therefore, it is possible to characterize \( \mathbf{\omega} \) as the angular velocity of a rotation motion, and \( \mathbf{\omega} \times \mathbf{r} = \mathbf{w} \cdot \mathbf{r} \) as the rotation velocity of the point that has \( \mathbf{r} \) as the position vector with respect to the rotation center (see Figure 2.21). Then, considering (2.90) and (2.92),

\[
d\mathbf{v} = l \cdot d\mathbf{x} = (d + \mathbf{w}) \cdot d\mathbf{x} = \underbrace{d \cdot d\mathbf{x}}_{\text{stretch velocity}} + \underbrace{\mathbf{w} \cdot d\mathbf{x}}_{\text{rotation velocity}},
\]

which allows describing the relative velocity \( d\mathbf{v} \) of the particles in the neighborhood of a given particle \( P \) (see Figure 2.22) as the sum of a relative stretch

\(^{23}\) Observe the similarity in the structure of tensors \( \mathbf{\Omega} \) and \( \mathbf{\theta} \) in Section 2.11.6 and of tensors \( \mathbf{w} \) and \( \mathbf{\omega} \) seen here.
velocity (characterized by the strain rate tensor $d$) and a relative rotation velocity (characterized by the spin tensor $w$ or the vorticity vector $2\omega$).

### 2.14 Material Time Derivatives of Strain and Other Magnitude Tensors

#### 2.14.1 Deformation Gradient Tensor and its Inverse Tensor

Differentiating the expression of $F$ in (2.3) with respect to time$^{24}$,

$$F_{ij} = \frac{\partial x_i(X,t)}{\partial X_j} \quad \Rightarrow \quad \frac{dF_{ij}}{dt} = \frac{\partial}{\partial t} \frac{\partial x_i(X,t)}{\partial X_j} = \frac{\partial}{\partial X_j} \left( \frac{\partial x_i(X,t)}{\partial t} \right) \Rightarrow (2.102)$$

$^{24}$ The Schwartz Theorem (equality of mixed partial derivatives) guarantees that for a function $\Phi(x_1, x_2, ..., x_n)$ that is continuous and has continuous derivatives, $\partial^2 \Phi/(\partial x_i \partial x_j) = \partial^2 \Phi/(\partial x_j \partial x_i)$ $\forall i, j$ is satisfied.
\[
\frac{dF_{ij}}{dt} = \frac{\partial v_i(X,t)}{\partial X_j} = \frac{\partial v_j(x(X,t))}{\partial x_i} \frac{\partial x_k}{\partial X_j} l_{ik} F_{kj} \implies \frac{d\mathbf{F}}{dt} = \dot{\mathbf{F}} = \mathbf{l} \cdot \mathbf{F}
\]

(2.102 (cont.))

where (2.91) has been taken into account for the velocity gradient tensor \( \mathbf{l} \). To obtain the material time derivative of tensor \( \mathbf{F}^{-1} \), the time derivative of the identity \( \mathbf{F} \cdot \mathbf{F}^{-1} = \mathbf{1} \) is performed\(^{25}\).

\[
\mathbf{F} \cdot \mathbf{F}^{-1} = \mathbf{1} \implies \frac{d}{dt} (\mathbf{F} \cdot \mathbf{F}^{-1}) = \frac{d\mathbf{F}}{dt} \cdot \mathbf{F}^{-1} + \mathbf{F} \cdot \frac{d(\mathbf{F}^{-1})}{dt} = 0
\]

\[
\implies \frac{d(\mathbf{F}^{-1})}{dt} = -\mathbf{F}^{-1} \cdot \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} - \mathbf{F}^{-1} \cdot \mathbf{F} \cdot \dot{\mathbf{F}}^{-1} = -\mathbf{F}^{-1} \cdot \mathbf{l}
\]

(2.103)

### 2.14.2 Material and Spatial Strain Tensors

From (2.21), (2.102) and (2.93), it follows\(^{26}\)

\[
\mathbf{E} = \frac{1}{2} (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1}) \implies \frac{d\mathbf{E}}{dt} = \dot{\mathbf{E}} = \frac{1}{2} \left( \dot{\mathbf{F}}^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \dot{\mathbf{F}} \right) = \frac{1}{2} \left( \mathbf{F}^T \cdot \mathbf{F} \cdot \dot{\mathbf{F}} + \dot{\mathbf{F}}^T \cdot \mathbf{F} \cdot \mathbf{F} \right) = \frac{1}{2} \mathbf{F}^T \cdot \left( \mathbf{I} + \dot{\mathbf{F}}^T \right) \cdot \mathbf{F} = \mathbf{F}^T \cdot \mathbf{d} \cdot \mathbf{F}
\]

\[
\implies \dot{\mathbf{E}} = \mathbf{F}^T \cdot \mathbf{d} \cdot \mathbf{F}
\]

(2.104)

\(^{25}\) The material time derivative of the inverse tensor \( d(\mathbf{F}^{-1})/dt \) must not be confused with the inverse of the material derivative of the tensor: \( (\dot{\mathbf{F}})^{-1} \). These two tensors are completely different tensors.

\(^{26}\) Observe that the result is the same as the one obtained in (2.98) using an alternative procedure.
Using (2.23) and (2.103) for the spatial strain tensor $\mathbf{e}$ yields

$$
\mathbf{e} = \frac{1}{2} (1 - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) \Rightarrow \frac{d\mathbf{e}}{dt} = \frac{\mathbf{e}}{2} \left( \frac{d}{dt} \left( \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} \right) + \mathbf{F}^{-T} \cdot \frac{d}{dt} \left( \mathbf{F}^{-1} \right) \right) =
$$

$$
\Rightarrow \frac{\mathbf{e}}{2} \left( \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} + \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} \cdot \mathbf{I} \right)
$$

(2.105)

2.14.3 Volume and Area Differentials

The volume differential $dV(\mathbf{X},t)$ associated with a certain particle $P$ varies along time (see Figure 2.23) and, in consequence, it makes sense to calculate its material derivative. Differentiating (2.55) for a volume differential results in

$$
dV(\mathbf{X},t) = |\mathbf{F}(\mathbf{X},t)| \ dV_0(\mathbf{X}) \Rightarrow \frac{d}{dt} dV(t) = \frac{d}{dt} |\mathbf{F}| dV_0 \ (2.106)
$$

Therefore, the material derivative of the determinant of the deformation gradient tensor $|\mathbf{F}|$ is\(^\text{27}\)

$$
\frac{d|\mathbf{F}|}{dt} = \frac{d|\mathbf{F}|}{dF_{ij}} \frac{dF_{ij}}{dt} = |\mathbf{F}| F_{ji}^{-1} l_{ik} = |\mathbf{F}| F_{ji}^{-1} l_{ik} \delta_{ij} = |\mathbf{F}| F_{ji}^{-1} l_{ij} = |\mathbf{F}| \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}
$$

(2.107)

where (2.102) and (2.91) have been considered. Introducing (2.107) into (2.106) and taking into account (2.55) finally results in

$$
\frac{d}{dt} (dV) = (\nabla \cdot \mathbf{v}) |\mathbf{F}| dV_0 = (\nabla \cdot \mathbf{v}) dV. \quad (2.108)
$$

Operating in a similar manner yields the material derivative of the area differential associated with a certain particle $P$ and a given direction $\mathbf{n}$ (see Figure 2.24). The area differential vector associated with a particle in the reference configuration, $d\mathbf{A}(\mathbf{X}) = d\mathbf{A}\mathbf{N}$, and in the present configuration, $d\mathbf{a}(\mathbf{x},t) = d\mathbf{a}\mathbf{n}$, are related through $d\mathbf{a} = |\mathbf{F}| \cdot d\mathbf{A} \cdot \mathbf{F}^{-1}$ (see (2.59)) and, differentiating this ex-

\(^{27}\) The derivative of the determinant of a tensor $\mathbf{A}$ with respect to the same tensor can be written in compact notation as $d|\mathbf{A}|/d\mathbf{A} = |\mathbf{A}| \cdot \mathbf{A}^{-T}$ or, in index notation, as $d|\mathbf{A}|/dA_{ij} = |\mathbf{A}| \cdot A_{ij}^{-1}$.

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pression, results in

\[ \frac{d}{dt} (da) = \frac{d}{dt} \left( |F| \cdot dA \cdot F^{-1} \right) = \frac{d}{dt} \left( |F| \nabla \cdot v \right) \frac{dA}{da} \cdot F^{-1} \cdot \left( \frac{d}{dt} \left( F^{-1} \right) \right) = \frac{d}{dt} \left( \left( \nabla \cdot v \right) |F| dA \cdot F^{-1} \right) = \frac{d}{dt} \left( \left( \nabla \cdot v \right) |F| \nabla \cdot dA \right) \]

where (2.103) and (2.107) have been considered.
2.15 Motion and Strains in Cylindrical and Spherical Coordinates

The expressions and equations obtained in intrinsic or compact notation are independent of the coordinate system considered. However, the expressions of the components depend on the coordinate system used. In addition to the Cartesian coordinate system, which has been used in the previous sections, two orthogonal curvilinear coordinate systems will be considered here: cylindrical coordinates and spherical coordinates.

Remark 2.23. An orthogonal curvilinear coordinate system (generically referred to as \( \{a, b, c\} \)), is characterized by its physical unit basis \( \{\hat{e}_a, \hat{e}_b, \hat{e}_c\} \) (\( \|\hat{e}_a\| = \|\hat{e}_b\| = \|\hat{e}_c\| = 1 \)), whose components are orthogonal to each other \( \hat{e}_a \cdot \hat{e}_b = \hat{e}_a \cdot \hat{e}_c = \hat{e}_b \cdot \hat{e}_c = 0 \)), as is also the case in a Cartesian system. The fundamental difference is that the orientation of the curvilinear basis changes at each point in space \( \hat{e}_m = \hat{e}_m(x) \quad m \in \{a, b, c\} \). Therefore, for the purposes here, an orthogonal curvilinear coordinate system can be considered as a mobile Cartesian coordinate system \( \{x', y', z'\} \) associated with a curvilinear basis \( \{\hat{e}_a, \hat{e}_b, \hat{e}_c\} \) (see Figure 2.25).

Remark 2.24. The components, of a certain magnitude of vectorial character \( \mathbf{v} \) or tensorial character \( \mathbf{T} \) in an orthogonal curvilinear coordinate system \( \{a, b, c\} \), can be obtained as the corresponding components in the local Cartesian system \( \{x', y', z'\} \):

\[
\mathbf{v}_n = \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \quad \mathbf{T}_n = \begin{bmatrix} T_{aa} & T_{ab} & T_{ac} \\ T_{ba} & T_{bb} & T_{bc} \\ T_{ca} & T_{cb} & T_{cc} \end{bmatrix} \equiv \begin{bmatrix} T_{x'x'} & T_{x'y'} & T_{x'z'} \\ T_{y'x'} & T_{y'y'} & T_{y'z'} \\ T_{z'x'} & T_{z'y'} & T_{z'z'} \end{bmatrix}
\]

Remark 2.25. The curvilinear components of the differential operators (the \( \nabla \) operator and its derivatives) are not the same as their counterparts in the local coordinate system \( \{x', y', z'\} \). They must be defined specifically for each case. Their value for cylindrical and spherical coordinates is provided in the corresponding section.
2.15.1 Cylindrical Coordinates

The position of a certain point in space can be defined by its cylindrical coordinates \( \{ r, \theta, z \} \) (see Figure 2.25). The figure also shows the physical orthonormal basis \( \{ \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z \} \). This basis changes at each point in space according to

\[
\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \quad \text{and} \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r.
\]

(2.110)

Figure 2.26 shows the corresponding differential element. The expressions in cylindrical coordinates of some of the elements treated in this chapter are:

- Nabla operator, \( \nabla \)

\[
\nabla = \frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{\partial}{\partial z} \mathbf{e}_z \quad \Rightarrow \quad \nabla \equiv \left[ \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right]^T
\]

(2.111)

\[
x(r, \theta, z) \equiv \begin{bmatrix} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{bmatrix}
\]

Figure 2.25: Cylindrical coordinates.

Figure 2.26: Differential element in cylindrical coordinates.
• **Displacement vector,** \( \mathbf{u} \), and **velocity vector,** \( \mathbf{v} \)

\[
\mathbf{u} = u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \hat{\mathbf{e}}_z \quad \implies \quad \mathbf{u} \equiv [u_r, u_\theta, u_z]^T
\]

\[
\mathbf{v} = v_r \hat{\mathbf{e}}_r + v_\theta \hat{\mathbf{e}}_\theta + v_z \hat{\mathbf{e}}_z \quad \implies \quad \mathbf{v} \equiv [v_r, v_\theta, v_z]^T
\]

(2.112)

(2.113)

• **Infinitesimal strain tensor,** \( \mathbf{\varepsilon} \)

\[
\mathbf{\varepsilon} = \frac{1}{2} \left( (\mathbf{u} \otimes \nabla) + (\mathbf{u} \otimes \nabla)^T \right) \equiv \begin{bmatrix}
\varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\
\varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\
\varepsilon_{rz} & \varepsilon_{\theta z} & \varepsilon_{zz}
\end{bmatrix}
\]

\[
\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \quad \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}
\]

\[
\varepsilon_{r\theta} = \frac{1}{r} \left( \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \quad \varepsilon_{r z} = \frac{1}{2} \left( \frac{\partial v_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)
\]

\[
\varepsilon_{\theta z} = \frac{1}{2} \left( \frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right)
\]

(2.114)

The components of \( \mathbf{\varepsilon} \) are presented on the corresponding differential element in Figure (2.26).

• **Strain rate tensor,** \( \mathbf{d} \)

\[
\mathbf{d} = \frac{1}{2} \left( (\mathbf{v} \otimes \nabla) + (\mathbf{v} \otimes \nabla)^T \right) \equiv \begin{bmatrix}
d_{rr} & d_{r\theta} & d_{r z} \\
d_{r\theta} & d_{\theta\theta} & d_{\theta z} \\
d_{r z} & d_{\theta z} & d_{zz}
\end{bmatrix}
\]

\[
d_{rr} = \frac{\partial v_r}{\partial r} \quad d_{\theta\theta} = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \quad d_{zz} = \frac{\partial v_z}{\partial z}
\]

\[
d_{r\theta} = \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \quad d_{r z} = \frac{1}{2} \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)
\]

\[
d_{\theta z} = \frac{1}{2} \left( \frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)
\]

(2.115)
2.15.2 Spherical Coordinates

A point in space is defined by its spherical coordinates \( \{r, \theta, \phi\} \). The physical orthonormal basis \( \{\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\} \) is presented in Figure 2.27. This basis changes at each point in space according to

\[
\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r, \quad \text{and} \quad \frac{\partial \hat{e}_\phi}{\partial \theta} = 0. \tag{2.116}
\]

The expressions in spherical coordinates of some of the elements treated in this chapter are:

- **Nabla operator, \( \nabla \)**

  \[
  \nabla = \frac{\partial}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{e}_\phi \implies \nabla \equiv \left[ \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right]^T \tag{2.117}
  \]

- **Displacement vector, \( u \), and velocity vector, \( v \)**

  \[
  u = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_\phi \hat{e}_\phi \implies u \equiv [u_r, u_\theta, u_\phi]^T \tag{2.118}
  \]

  \[
  v = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_\phi \hat{e}_\phi \implies v \equiv [v_r, v_\theta, v_\phi]^T \tag{2.119}
  \]

\[x(r, \theta, \phi) \equiv \begin{bmatrix} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{bmatrix}\]

Figure 2.27: Spherical coordinates.
• **Infinitesimal strain tensor, \( \varepsilon \)**

\[
\varepsilon = \frac{1}{2} \left( (u \otimes \nabla) + (u \otimes \nabla)^T \right) = \begin{bmatrix}
\varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{r\phi} \\
\varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta\phi} \\
\varepsilon_{r\phi} & \varepsilon_{\theta\phi} & \varepsilon_{\phi\phi}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right) \\
\frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_{\phi}}{\partial r} - \frac{u_{\phi}}{r} \right) \\
\frac{1}{2} \left( \frac{1}{r \sin \theta} \frac{\partial u_{\theta}}{\partial \phi} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{r \cot \phi} \right)
\end{bmatrix}
\]

The components of \( \varepsilon \) are presented on the corresponding differential element in Figure 2.28.

• **Strain rate tensor, \( d \)**

\[
d = \frac{1}{2} \left( (v \otimes \nabla) + (v \otimes \nabla)^T \right) = \begin{bmatrix}
d_{rr} & d_{r\theta} & d_{r\phi} \\
d_{r\theta} & d_{\theta\theta} & d_{\theta\phi} \\
d_{r\phi} & d_{\theta\phi} & d_{\phi\phi}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \left( \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \\
\frac{1}{2} \left( \frac{1}{r} \frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} \right) \\
\frac{1}{2} \left( \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} - \frac{v_\phi}{r \cot \phi} \right)
\end{bmatrix}
\]

(2.120)
Figure 2.28: Differential element in spherical coordinates.

\[ dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \]
PROBLEMS

Problem 2.1 – A deformation that takes place in a continuous medium has the following consequences on the triangle shown in the figure below:

1. The segment $OA$ increases its initial length in $(1 + p)$.
2. The angle $AOB$ decreases in $q$ radians its initial value.
3. The area increases its initial value in $(1 + r)$.
4. $p, q, r, s \ll 1$.

The deformation is uniform and the $z$-axis is one of the principal directions of the deformation gradient tensor, which is symmetric. In addition, the stretch in this direction is known to be $\lambda_z = 1 + s$. Obtain the infinitesimal strain tensor.

Solution

A uniform deformation implies that the deformation gradient tensor ($F$) does not depend on the spatial variables. Consequently, the strain tensor ($E$) and the stretches ($\lambda$) do not depend on them either. Also, note that the problem is to be solved under infinitesimal strain theory.

The initial and final lengths of a segment parallel to the $x$-axis are related as follows.

$$\frac{OA_{final}}{OA_{initial}} = \left( \int_{O}^{A} \lambda_x dX = \lambda_x \int_{O}^{A} dX = \lambda_x OA_{initial} \right) \left\{ \begin{array}{c} \Rightarrow \lambda_x = 1 + p \end{array} \right.$$
Also, an initial right angle (the angle between the $x$- and $y$-axes) is related to its corresponding final angle after the deformation through

$$\text{initial angle } = \frac{\pi}{2} \quad \Rightarrow \quad \Delta \Phi_{xy} = -\gamma_{xy} = -2\varepsilon_{xy} = -q \quad \Rightarrow \quad \varepsilon_{xy} = \frac{q}{2}.$$ 

In addition, $\mathbf{F}$ is symmetric and the $z$-axis is a principal direction, therefore

$$\mathbf{F} \neq \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{12} & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix} \neq \begin{bmatrix} 1 & \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_x}{\partial y} & 1 & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_x}{\partial y} & 1 \end{bmatrix} + \begin{bmatrix} \lambda^2 \\ \lambda^2 \\ \lambda^2 \end{bmatrix},$$

which reveals the nature of the components of the displacement vector,

$$\begin{cases} \frac{\partial u_x}{\partial z} = \frac{\partial u_y}{\partial z} = 0 \\ \frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} u_x(x,y) \\ u_y(x,y) \\ u_z(z) \end{cases}.$$ 

Then, the following components of the strain tensor can be computed.

$$\varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = 0 \quad \Rightarrow \quad \varepsilon_{xz} = 0$$

$$\varepsilon_{zx} = \frac{1}{2} \left( \frac{\partial u_z}{\partial z} + \frac{\partial u_x}{\partial x} \right) = 0 \quad \Rightarrow \quad \varepsilon_{zx} = 0$$

$$\varepsilon_{zz} = \frac{1}{2} \left( \frac{\partial u_z}{\partial z} \right) = \lambda - 1 \quad \Rightarrow \quad \varepsilon_{zz} = s$$

In infinitesimal strain theory, $\mathbf{F} = 1 + \varepsilon + \Omega$, where $\Omega_{33} = 0$ since the infinitesimal rotation tensor is antisymmetric. Thus, $F_{zz} = 1 + \varepsilon_{zz}$ results in $F_{zz} = 1 + s$.

Now, the relation between the initial and final areas is $dA = |\mathbf{F}| dA_0 \cdot \mathbf{F}^{-1}$, where the inverse tensor of $\mathbf{F}$ is calculated using the notation

$$\mathbf{F}^{-1} \equiv \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & 1+s \end{bmatrix} \quad \text{with} \quad \mathbf{B}^{-1} \equiv \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix},$$

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which yields the inverse tensor of $F$,

$$
F^{-1} \equiv \begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & 1 + s
\end{bmatrix}.
$$

The area differential vector is defined as

$$
dA_0 \not\equiv \begin{bmatrix}
0 \\
0 \\
dA_0
\end{bmatrix} \implies dA_0 \cdot F^{-1} \not\equiv \begin{bmatrix}
0 \\
0 \\
1 + s dA_0
\end{bmatrix}.
$$

Then, taking into account that $|F| = \text{Tr}(\varepsilon) + 1$, and neglecting second-order terms results in

$$
dA = (1 + r) dA_0 \\
dA = (1 + p + s + \varepsilon_{yy}) \frac{1}{1 + s} dA_0
$$

Finally, since the strain tensor is symmetric,

$$
\varepsilon \equiv \begin{bmatrix}
p & \frac{q}{2} & 0 \\
\frac{q}{2} & r - p & 0 \\
0 & 0 & s
\end{bmatrix}.
$$

**Problem 2.2** – A uniform deformation ($F = F(t)$) is produced on the tetrahedron shown in the figure below, with the following consequences:

1. Points $O$, $A$ and $B$ do not move.
2. The volume of the solid becomes $p$ times its initial volume.
3. The length of segment $\overline{AC}$ becomes $p/\sqrt{2}$ times its initial length.
4. The final angle $AOC$ has a value of $45^\circ$. 

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Then,

a) Justify why the infinitesimal strain theory cannot be used here.

b) Determine the deformation gradient tensor, the possible values of $p$ and the displacement field in its material and spatial forms.

c) Draw the deformed solid.

Solution

a) The angle $\angle AOC$ changes from $90^\circ$ to $45^\circ$ therefore, it is obvious that the deformation involved is not infinitesimal. In addition, under infinitesimal strain theory $\Delta \Phi \ll 1$ is satisfied and, in this problem, $\Delta \Phi = \pi/4 \approx 0.7854$.

Observation: strains are dimensionless; in engineering, small strains are usually considered when these are of order $10^{-3} - 10^{-4}$.

b) The conditions in the statement of the problem must be imposed one by one:

1. Considering that $F(X, t) = F(t)$ and knowing that $dx = F \cdot dX$, the latter can be integrated as

$$x = \int dx = \int F dX = F \int dX = F(t) \cdot X + C(t)$$

with $F \equiv \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}$ and $C \equiv \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$, which results in 12 unknowns. Imposing now the conditions in the statement,

Point $O$ does not move:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F \\ F \cdot dX \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + C \Rightarrow C \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Point $A$ does not move:

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a F_{11} \\ a F_{21} \\ a F_{31} \end{bmatrix} \Rightarrow \begin{cases} F_{11} = 1 \\ F_{21} = 0 \\ F_{31} = 0 \end{cases}$$
Point $B$ does not move:

$$
\begin{bmatrix}
0 \\
a \\
0
\end{bmatrix} = \begin{bmatrix} aF_{12} \\
F_{22} \\
aF_{32}
\end{bmatrix} = \begin{bmatrix} F_{12} = 0 \\
F_{22} = 1 \\
F_{32} = 0
\end{bmatrix}
$$

Grouping all the information obtained results in

$$F_{12} = 0 \quad F_{22} = 1 \quad F_{32} = 0$$

2. The condition in the statement imposes that $V_{\text{final}} = pV_{\text{initial}}$.
Expression $dV_f = |F| dV_0$ allows to locally relate the differential volumes at different instants of time. In this case, $F$ is constant for each fixed $t$, thus, the expression can be integrated and the determinant of $F$ can be moved outside the integral,

$$V_f = \int_V dV_f = \int_{V_0} |F| dV_0 = |F| \int_{V_0} dV_0 = |F| V_0 .$$

Therefore, $|F| = F_{33} = p$ must be imposed.

3. The condition in the statement imposes that $l_{AC,\text{final}} = p \sqrt{2} l_{AC,\text{initial}}$.
Since $F$ is constant, the transformation is linear, that is, it transforms straight lines into straight lines. Hence, $AC$ in the deformed configuration must also be a rectilinear segment. Then,

$$x_C = F \cdot x_C = \begin{bmatrix} 1 & 0 & F_{13} \\
0 & 1 & F_{23} \\
0 & 0 & F_{33}
\end{bmatrix} \begin{bmatrix} 0 \\
a \\
a
\end{bmatrix} = \begin{bmatrix} aF_{13} \\
aF_{23} \\
ap
\end{bmatrix}$$

and

$$l_{AC,\text{final}} = l_{AC} = |[aF_{13}, aF_{23}, ap] - [a, 0, 0]| = |a(F_{13} - 1), aF_{23}, ap| =
\sqrt{(a(F_{13} - 1))^2 + (aF_{23})^2 + (ap)^2} = a \sqrt{(F_{13} - 1)^2 + F_{23}^2 + p^2} =
\frac{p}{\sqrt{2}} l_{AC} = \frac{p}{\sqrt{2}} \sqrt{2} a = pa .$$

Therefore,

$$\sqrt{(F_{13} - 1)^2 + F_{23}^2 + p^2} = p \quad \Rightarrow \quad (F_{13} - 1)^2 + F_{23}^2 = 0 \quad \Rightarrow \quad F_{13} = 1; \ F_{23} = 0$$
and the deformation gradient tensor results in

\[
F \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix},
\]

such that only the value of \( p \) remains to be found.

4. The condition in the statement imposes that \( AOC_{\text{final}} = 45^\circ = \pi / 4 \).

Considering \( d\mathbf{X}^{(1)} \equiv [1, 0, 0] \) and \( d\mathbf{X}^{(2)} \equiv [0, 0, 1] \), the corresponding vectors in the spatial configuration are computed as

\[
d\mathbf{x}^{(1)} = F \cdot d\mathbf{X}^{(1)} \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

\[
d\mathbf{x}^{(2)} = F \cdot d\mathbf{X}^{(2)} \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}.
\]

Then,

\[
\cos (AOC_{\text{final}}) = \cos 45^\circ = \frac{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}}{|d\mathbf{x}^{(1)}| |d\mathbf{x}^{(2)}|} = \frac{\sqrt{2}}{2}
\]

is imposed, with

\[
|d\mathbf{x}^{(1)}| = 1, \quad |d\mathbf{x}^{(2)}| = \sqrt{1 + p^2} \quad \text{and} \quad d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = 1
\]

such that

\[
\frac{1}{\sqrt{1 + p^2}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \implies p = \pm 1.
\]

But \(|F| = p > 0\), and, consequently, \( p = 1 \). Then, the deformation gradient tensor is

\[
F \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
The equation of motion is determined by means of \( \mathbf{x} = \mathbf{F} \cdot \mathbf{X} \),

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 1 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix} =
\begin{bmatrix}
  X + Z \\
  Y \\
  Z
\end{bmatrix},
\]

which allows determining the displacement field in material and spatial descriptions as

\[
\mathbf{U}(\mathbf{X}, t) = \mathbf{x} - \mathbf{X} \equiv 
\begin{bmatrix}
  Z \\
  0 \\
  0
\end{bmatrix}
\quad \text{and} \quad
\mathbf{u}(\mathbf{x}, t) \equiv 
\begin{bmatrix}
  Z \\
  0 \\
  0
\end{bmatrix}.
\]

c) The graphical representation of the deformed tetrahedron is:

**Problem 2.3** – A uniform deformation is applied on the solid shown in the figure below. Determine:

a) The general expression of the material description of the displacement field \( \mathbf{U}(\mathbf{X}, t) \) in terms of the material displacement gradient tensor \( \mathbf{J} \).

b) The expression of \( \mathbf{U}(\mathbf{X}, t) \) when, in addition, the following boundary conditions are satisfied:

\[
\begin{align*}
U_Y &= U_Z = 0, \quad \forall X, Y, Z \\
U_X \bigg|_{X=0} &= 0, \quad \forall X, Y \\
U_X \bigg|_{X=L} &= \delta
\end{align*}
\]
c) The possible values (positive and negative) that $\delta$ may take. Justify the answer obtained.

d) The material and spatial strain tensors and the infinitesimal strain tensor.

e) Plot the curves $E_{XX} - \delta/L$, $e_{xx} - \delta/L$ and $\varepsilon_x - \delta/L$ for all possible values of $\delta$, indicating every significant value.

Solution

a) A uniform deformation implies that $F(X,t) = F(t), \forall X$. The deformation gradient tensor is related to the material displacement gradient tensor through the expression $F = I + J$. Therefore, if $F = F(t)$, then $J = J(t)$. Taking into account the definition of $J$ and integrating its expression results in

$$ J = \frac{\partial U(X,t)}{\partial X} \implies dU = J dX \implies \int dU = \int J dX $$

$$ \implies \int dU = J \int dX \implies U = J \cdot X + C(t). $$

where $C(t)$ is an integration constant. Then, the general expression of the material description of the displacement field is

$$ U(X,t) = J(t) \cdot X + C(t). $$

b) Using the previous result and applying the boundary conditions given in the statement of the problem will yield the values of $J$ and $C$.

Boundary conditions:

$$ U_Y = U_Z = 0, \forall X, Y, Z \implies \text{Points only move in the X-direction.} $$

$$ U_X|_{X=0} = 0, \forall Y, Z \implies \text{The YZ plane at the origin is fixed.} $$

$$ U_X|_{X=L} = \delta, \forall Y, Z \implies \text{This plane moves in a uniform manner in the X-direction.} $$
If the result obtained in a) is written in component form, the equations and conclusions that can be reached will be understood better.

\[ U_X = J_{11}X + J_{12}Y + J_{13}Z + C_1 \]
\[ U_Y = J_{21}X + J_{22}Y + J_{23}Z + C_2 \]
\[ U_Z = J_{31}X + J_{32}Y + J_{33}Z + C_3 \]

From the first boundary condition:

\[ U_Y = 0, \quad \forall X, Y, Z \quad \implies \quad J_{21} = J_{22} = J_{23} = C_2 = 0 \]
\[ U_Z = 0, \quad \forall X, Y, Z \quad \implies \quad J_{31} = J_{32} = J_{33} = C_3 = 0 \]

From the second boundary condition:

\[ U_X|_{X=0} = 0, \quad \forall Y, Z \quad \implies \quad J_{12} = J_{13} = C_1 = 0 \]

From the third boundary condition:

\[ U_X|_{X=L} = \delta, \quad \forall Y, Z \quad \implies \quad J_{11}L = \delta \quad \implies \quad J_{11} = \frac{\delta}{L} \]

Finally,

\[ J \equiv \begin{bmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \implies \quad U(X) = J \cdot X + C \equiv \begin{bmatrix} \delta X \\ \frac{\delta}{L} \\ 0 \end{bmatrix} \]

\( c) \) In order to justify all the possible positive and negative values that \( \delta \) may take, the condition \( |F| > 0 \) must be imposed. Therefore, the determinant of \( F \) must be computed,

\[ F = 1 + J \equiv \begin{bmatrix} 1 + \frac{\delta}{L} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \implies \quad |F| = 1 + \frac{\delta}{L} > 0 \quad \implies \quad \delta > -L \]

\( d) \) To obtain the spatial and material strain tensors as well as the infinitesimal strain tensor, their respective definitions must be taken into account.

Spatial strain tensor:

\[ e = \frac{1}{2} (1 - F^{-T} \cdot F^{-1}) \]
Material strain tensor:  \[ \mathbf{E} = \frac{1}{2} (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) \]

Infinitesimal strain tensor:  \[ \mathbf{\varepsilon} = \frac{1}{2} (\mathbf{J}^T \cdot \mathbf{J}) \]

Applying these definitions using the values of \( \mathbf{F} \) and \( \mathbf{J} \) calculated in \( b) \) and \( c) \), the corresponding expressions are obtained.

\[
\mathbf{e} = \begin{bmatrix} e_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad e_{xx} = \left( \frac{\delta}{L} + \frac{1}{2} \frac{\delta^2}{L^2} \right) \left( 1 + \frac{\delta}{L} \right)^2
\]

\[
\mathbf{E} = \begin{bmatrix} E_{XX} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad E_{XX} = \frac{\delta}{L} + \frac{1}{2} \frac{\delta^2}{L^2}
\]

\[
\mathbf{\varepsilon} = \begin{bmatrix} \delta \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[ e) \quad \text{Plotting the curves } E_{XX} - \frac{\delta}{L}, \quad e_{xx} = \frac{\delta}{L} \quad \text{and} \quad \varepsilon_x = \frac{\delta}{L} \text{ together yields:}
\]

Here,

- \( E_{XX} \) is a second-order parabola that contains the origin and has its minimum at \( \frac{\delta}{L} = -1 \), i.e., for \( E_{XX} = -1/2 \).
- \( \varepsilon_x \) is the identity straight line (45° slope and contains the origin).
- \( e_{xx} \) has two asymptotes, a vertical one at \( \frac{\delta}{L} = -1 \) and a horizontal at \( e_{xx} = 1/2 \).

It can be concluded, then, that for small \( \frac{\delta}{L} \) strains the three functions have a very similar behavior and the same slope at the origin. That is, the same result will be obtained with any of the definitions of strain tensor. However, outside this domain (large or finite strains) the three curves are clearly different.
EXERCISES

2.1 – Consider the velocity fields
\[
v_1 \equiv \begin{bmatrix} \frac{x}{1+t} \\ \frac{2y}{1+t} \\ \frac{3z}{1+t} \end{bmatrix}^T \quad \text{and} \quad v_2 \equiv \begin{bmatrix} \frac{X}{1+t} \\ \frac{2Y}{1+t} \\ \frac{3Z}{1+t} \end{bmatrix}^T.
\]
Determine:

a) The material description of \( v_1 \) and the spatial description of \( v_2 \) (consider \( t = 0 \) is the reference configuration).

b) The density distribution in both cases (consider \( \rho_0 \) is the initial density).

c) The material and spatial descriptions of the displacement field as well as the material (Green-Lagrange) and spatial (Almansi) strain tensors for the velocity field \( v_1 \).

d) Repeat c) for configurations close to the reference configuration \( (t \to 0) \).

e) Prove that the two strain tensors coincide for the conditions stated in d).

2.2 – The equation of motion in a continuous medium is
\[
x = X + Y t, \quad y = Y, \quad z = Z.
\]
Obtain the length at time \( t = 2 \) of the segment of material line that at time \( t = 1 \) is defined in parametric form as
\[
x(\alpha) = 0, \quad y(\alpha) = \alpha^2, \quad z(\alpha) = \alpha \quad 0 \leq \alpha \leq 1.
\]

2.3 – Consider the material strain tensor
\[
E \equiv \begin{bmatrix} 0 & te^{\epsilon X} & 0 \\ te^{\epsilon X} & 0 & 0 \\ 0 & 0 & te^{\epsilon Y} \end{bmatrix}.
\]
Obtain the length at time \( t = 1 \) of the segment that at time \( t = 0 \) (reference configuration) is straight and joins the points \((1,1,1)\) and \((2,2,2)\).
2.4 – The equation of motion of a continuous medium is

\[ x = X, \quad y = Y, \quad z = Z - Xt. \]

Calculate the angle formed at time \( t = 0 \) by the differential segments that at time \( t = t \) are parallel to the \( x \)- and \( z \)-axes.

2.5 – The following information is known in relation to a certain displacement field given in material description, \( U(X, Y, Z) \):
1) It is linear in \( X, Y, Z \).
2) It is antisymmetric with respect to plane \( Y = 0 \), that is, the following is satisfied:

\[ U(X, Y, Z) = -U(X, -Y, Z) \]

\[ \forall X, Y, Z \]

3) Under said displacement field, the volume of the element in the figure does not change, its angle \( AOB \) remains constant, the segment \( OB \) becomes \( \sqrt{2} \) times its initial length and the vertical component of the displacement at point \( B \) is positive (\( w_B > 0 \)).

Determine:

a) The most general expression of the given displacement field, such that conditions 1) and 2) are satisfied.

b) The expression of \( U \) when, in addition, condition 3) is satisfied. Obtain the deformation gradient tensor and the material strain tensor. Draw the deformed shape of the element in the figure, indicating the most significant values.

c) The directions (defined by their unit vectors \( \mathbf{T} \)) for which the deformation is reduced to a stretch (there is no rotation).

NOTE: Finite strains must be considered (not infinitesimal ones).
2.6 – The solid in the figure undergoes a uniform deformation such that points A, B and C do not move. Assuming an infinitesimal strain framework,

a) Express the displacement field in terms of “generic” values of the stretches and rotations.

b) Identify the null components of the strain tensor and express the rotation vector in terms of the stretches.

In addition, the following is known:
1) Segment $\overline{AE}$ becomes $(1+p)$ times its initial length.
2) The volume becomes $(1+q)$ times its initial value.
3) The angle $\theta$ increases its value in $r$ (given in radians).

Under these conditions, determine:

c) The strain tensor, the rotation vector and the displacement field in terms of $p$, $q$ and $r$.

NOTE: The values of $p$, $q$ and $r$ are small and its second-order infinitesimal terms can be neglected.

2.7 – The solid in the figure undergoes a uniform deformation with the following consequences:
1) The $x$- and $z$-axes are both material lines. Point A does not move.
2) The volume of the solid remains constant.
3) The angle $\theta_{xy}$ remains constant.
4) The angle $\theta_{yz}$ increases in $r$ radians.
5) The segment $\overline{AF}$ becomes $(1+p)$ times its initial length.
6) The area of the triangle $\triangle ABE$ becomes $(1+q)$ its initial value.
2.8 – The sphere in the figure undergoes a uniform deformation ($F = \text{const.}$) such that points A, B and C move to positions $A'$, $B'$ and $C'$, respectively. Point O does not move. Determine:

a) The deformation gradient tensor in terms of $p$ and $q$.

b) The equation of the deformed external surface of the sphere. Indicate which type of surface it is and draw it.

c) The material and spatial strain tensors. Obtain the value of $p$ in terms of $q$ when the material is assumed to be incompressible.

d) Repeat c) using infinitesimal strain theory. Prove that when $p$ and $q$ are small, the results of c) and d) coincide.

NOTE: The values of $p$, $q$ and $r$ are small and its second-order infinitesimal terms can be neglected.