Multimedia course

CONTINUUM MECHANICS FOR ENGINEERS

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The available media

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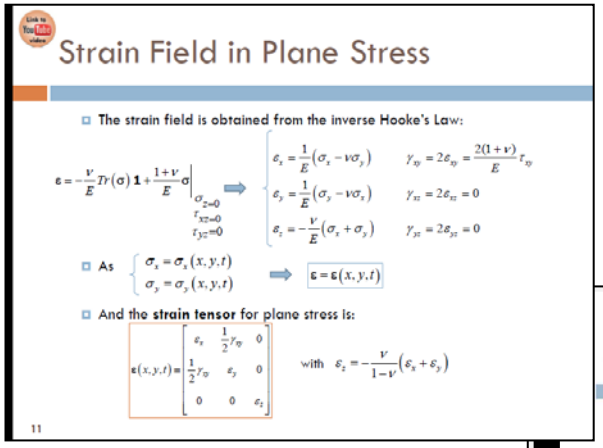

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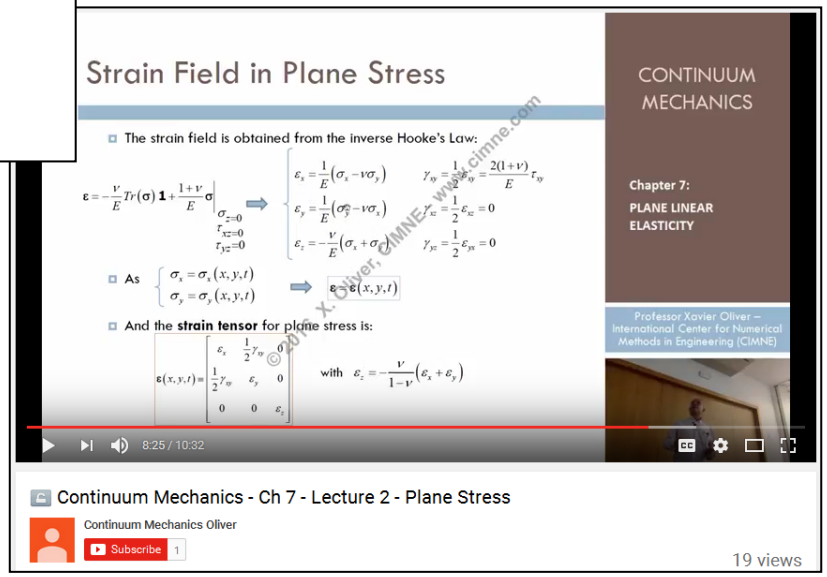

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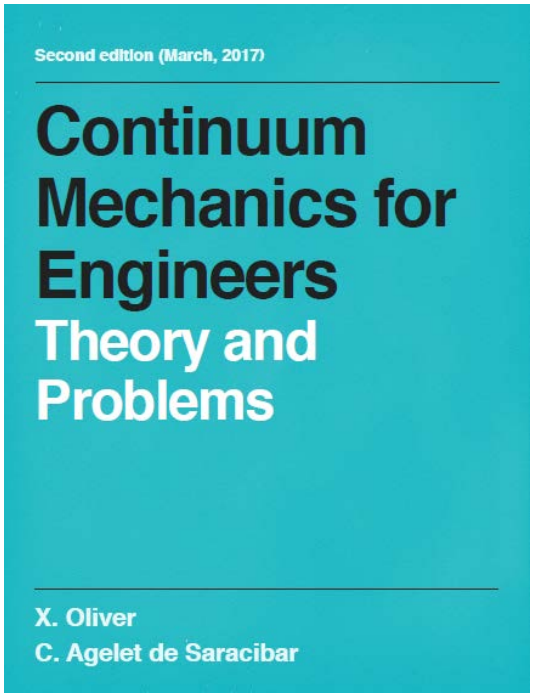

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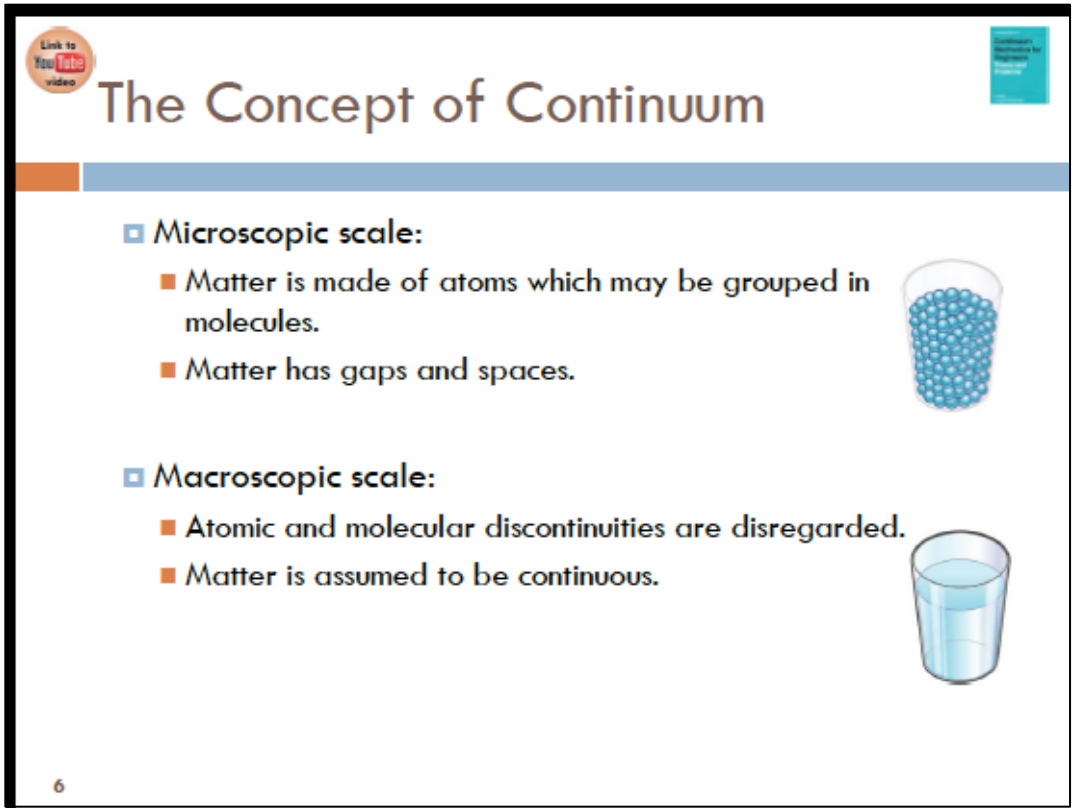

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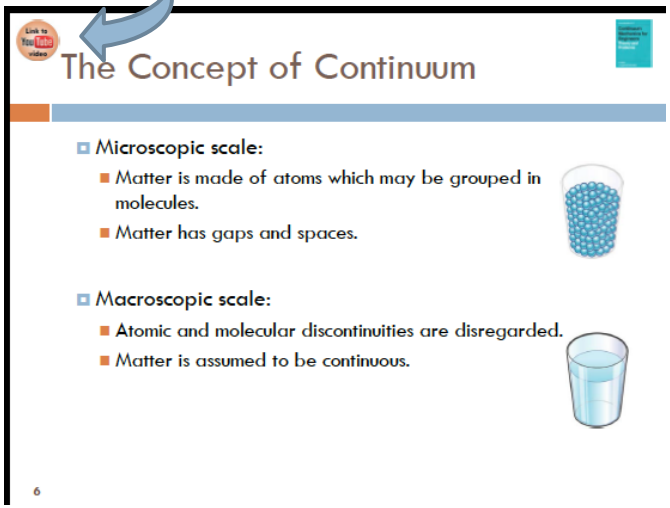

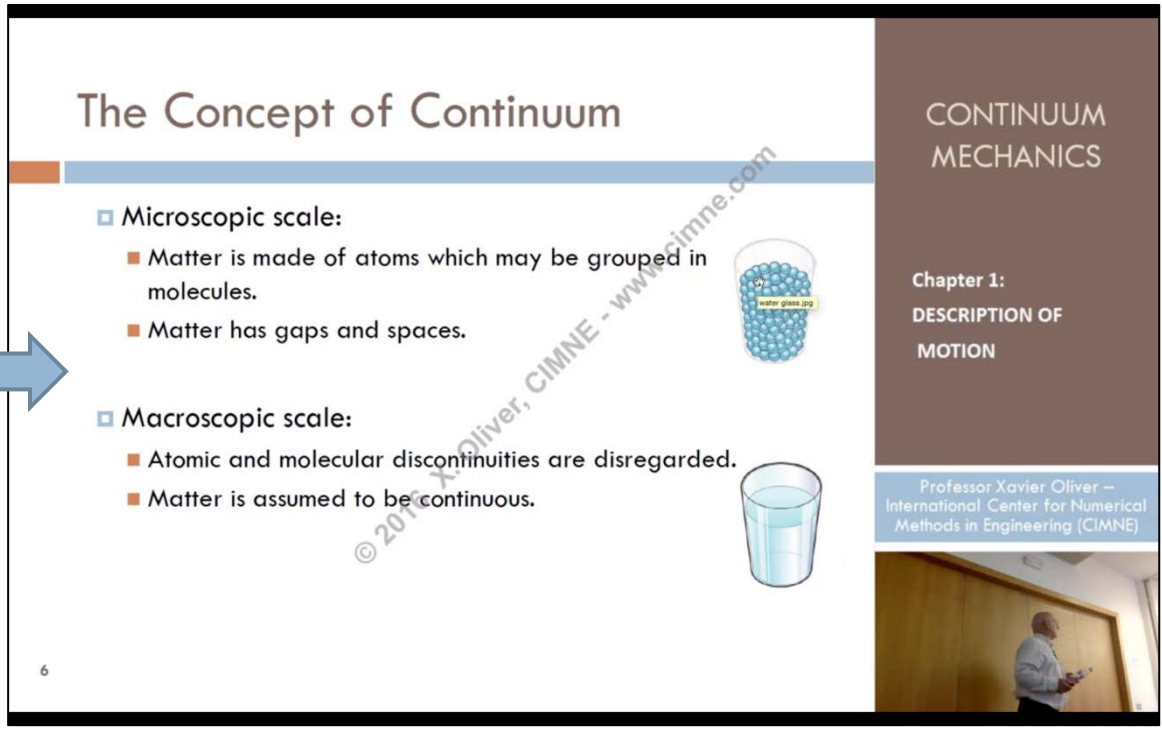

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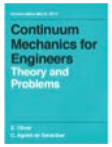
③ To reach the book-section dealing with the slide, click on the **link-to-book** icon 

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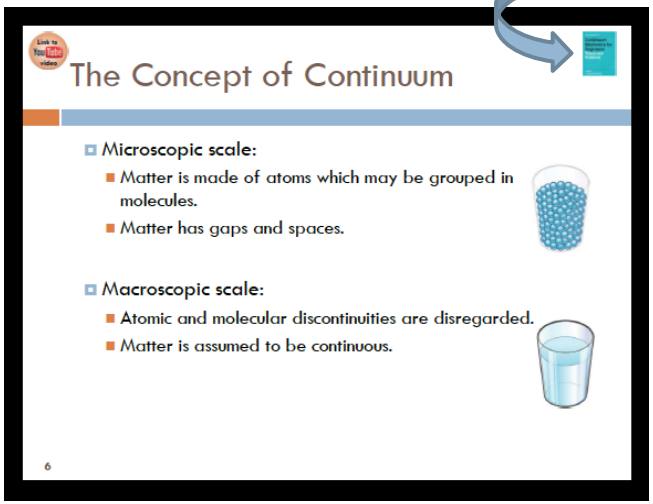
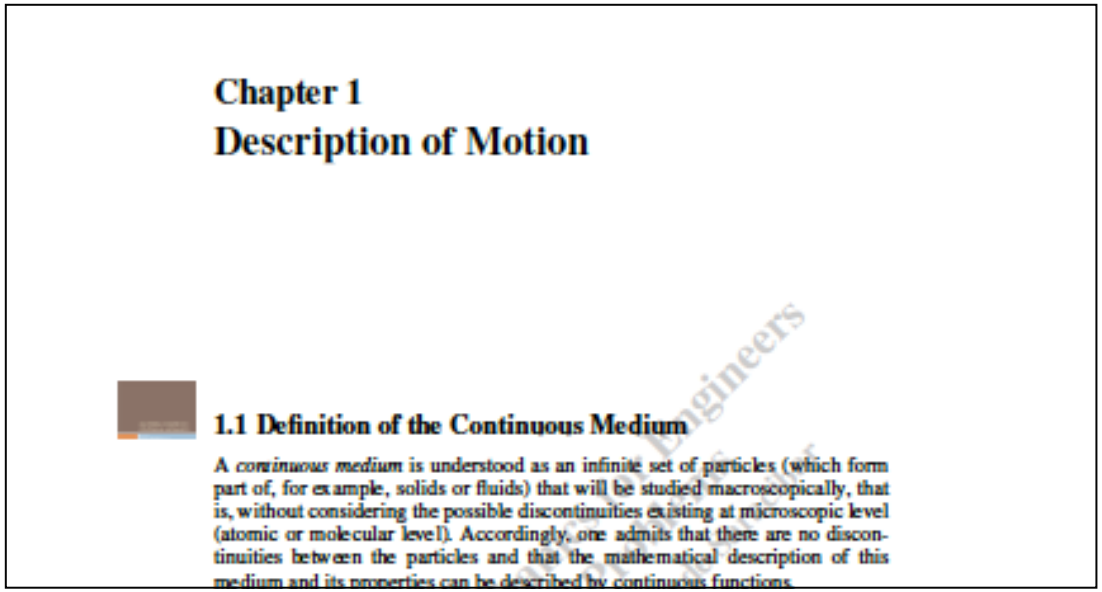




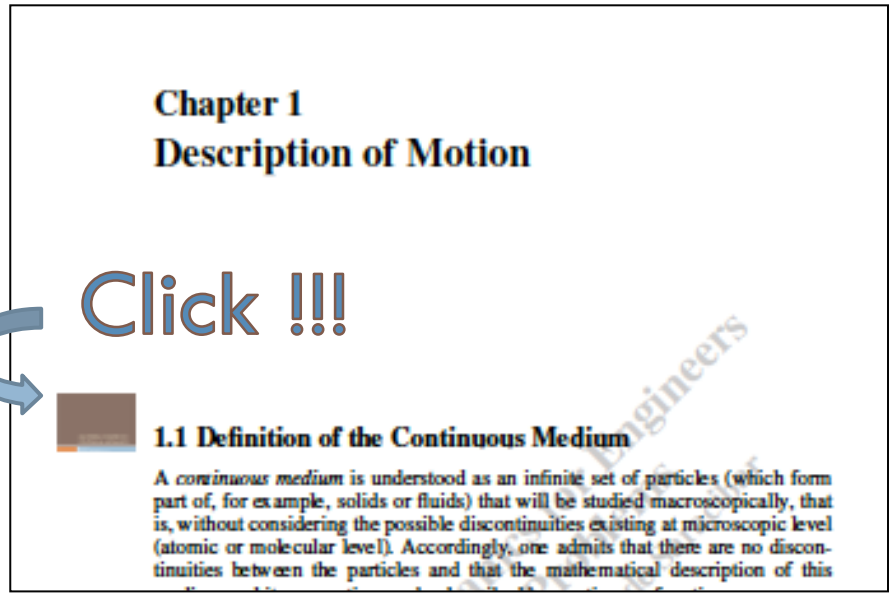
Chapter 1
Description of Motion

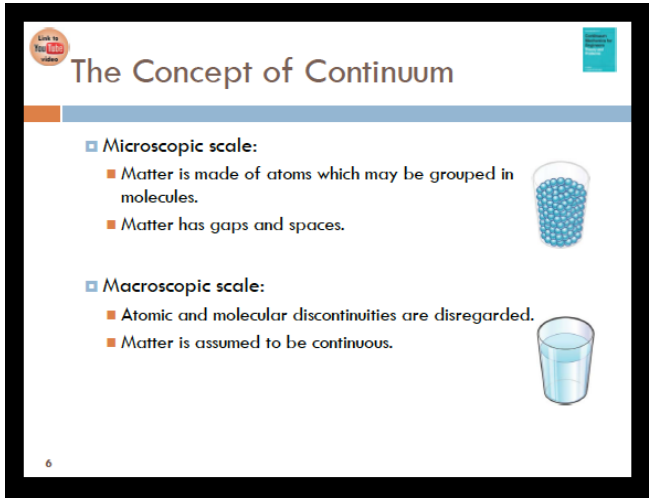
1.1 Definition of the Continuous Medium

A *continuous medium* is understood as an infinite set of particles (which form part of, for example, solids or fluids) that will be studied macroscopically, that is, without considering the possible discontinuities existing at microscopic level (atomic or molecular level). Accordingly, one admits that there are no discontinuities between the particles and that the mathematical description of this medium and its properties can be described by continuous functions.

How to interact with the media ...

4 To return to the slide, click on the **link-to-slide** icon





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Thanks for your contribution !!!!

CH.1. DESCRIPTION OF MOTION

Multimedia Course on Continuum Mechanics

Overview

1.1. Definition of the Continuous Medium

- 1.1.1. Concept of Continuum
- 1.1.2. Continuous Medium or Continuum

Lecture 1 Link to YouTube video

1.2. Equations of Motion

- 1.2.1 Configurations of the Continuous Medium
- 1.2.2. Material and Spatial Coordinates
- 1.2.3. Equation of Motion and Inverse Equation of Motion
- 1.2.4. Mathematical Restrictions

Lecture 2 Link to YouTube video

- 1.2.5. Example

Lecture 3 Link to YouTube video

1.3. Descriptions of Motion

- 1.3.1. Material or Lagrangian Description
- 1.3.2. Spatial or Eulerian Description

Lecture 4 Link to YouTube video

- 1.3.3. Example

Lecture 5 Link to YouTube video

Overview (cont'd)

1.4. Time Derivatives

- 1.4.1. Material and Local Derivatives

Lecture 6 — Link to YouTube video

- 1.4.2. Convective Rate of Change

Lecture 7 — Link to YouTube video

- 1.4.3. Example

Lecture 8 — Link to YouTube video

1.5. Velocity and Acceleration

- 1.5.1. Velocity
- 1.5.2. Acceleration
- 1.5.3. Example

Lecture 9 — Link to YouTube video

1.6. Stationarity and Uniformity

- 1.6.1. Stationary Properties
- 1.6.2. Uniform Properties

Lecture 10 — Link to YouTube video

Overview (cont'd)

1.7. Trajectory or Pathline

- 1.7.1. Equation of the Trajectories
- 1.7.2. Example

1.8. Streamlines

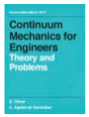
- 1.8.1. Equation of the Streamlines
- 1.8.2. Trajectories and Streamlines
- 1.8.3. Example
- 1.8.4. Streamtubes

Lecture 11 Link to YouTube video

1.9. Control and Material Surfaces

- 1.9.1. Control Surface
- 1.9.2. Material Surface
- 1.9.3. Control Volume
- 1.9.4. Material Volume

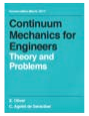
Lecture 12 Link to YouTube video



1.1 Definition of the Continuous Medium

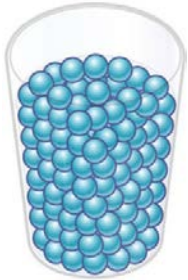
Ch.1. Description of Motion

Link to YouTube video



The Concept of Continuum

- Microscopic scale:
  - Matter is made of atoms which may be grouped in molecules.
  - Matter has gaps and spaces.



- Macroscopic scale:
  - Atomic and molecular discontinuities are disregarded.
  - Matter is assumed to be continuous.

Continuous Medium or Continuum

- Matter is studied at a macroscopic scale: it <u>completely fills the space</u>, there exist no gaps or empty spaces.

- Assumption that the medium and is made of infinite particles (of infinitesimal size) whose properties are describable by <u>continuous functions</u> with continuous derivatives.

Exceptions to the Continuous Medium

- Exceptions will exist where the theory will not account for all the observed properties of matter. E.g.: fatigue cracks.
  - In occasions, continuum theory can be used in combination with empirical information or information derived from a physical theory based on the molecular nature of material.

- The existence of areas in which the theory is not applicable does not destroy its usefulness in other areas.

Continuum Mechanics

- Study of the mechanical behavior of a continuous medium when subjected to forces or displacements, and the subsequent effects of this medium on its environment.

- It divides into:
  - **General Principles**: assumptions and consequences applicable to all continuous media.
  - **Constitutive Equations**: define the mechanical behavior of a particular idealized material.

1.2 Equations of Motion

Ch.1. Description of Motion

Material and Spatial points, Configuration

- A continuous medium is formed by an infinite number of particles which occupy different positions in space during their movement over time.
  - **MATERIAL POINTS**: particles
  - **SPATIAL POINTS**: fixed spots in space

- The **CONFIGURATION** Ω_t of a continuous medium at a given time (t) is the locus of the positions occupied by the material points of the continuous medium at the given time.

Configurations of the Continuous Medium

Ω_0: non-deformed (or reference) configuration, at reference time t_0.

Γ_0 : non-deformed boundary.

$\boldsymbol{X}$: Position vector of a particle at reference time.

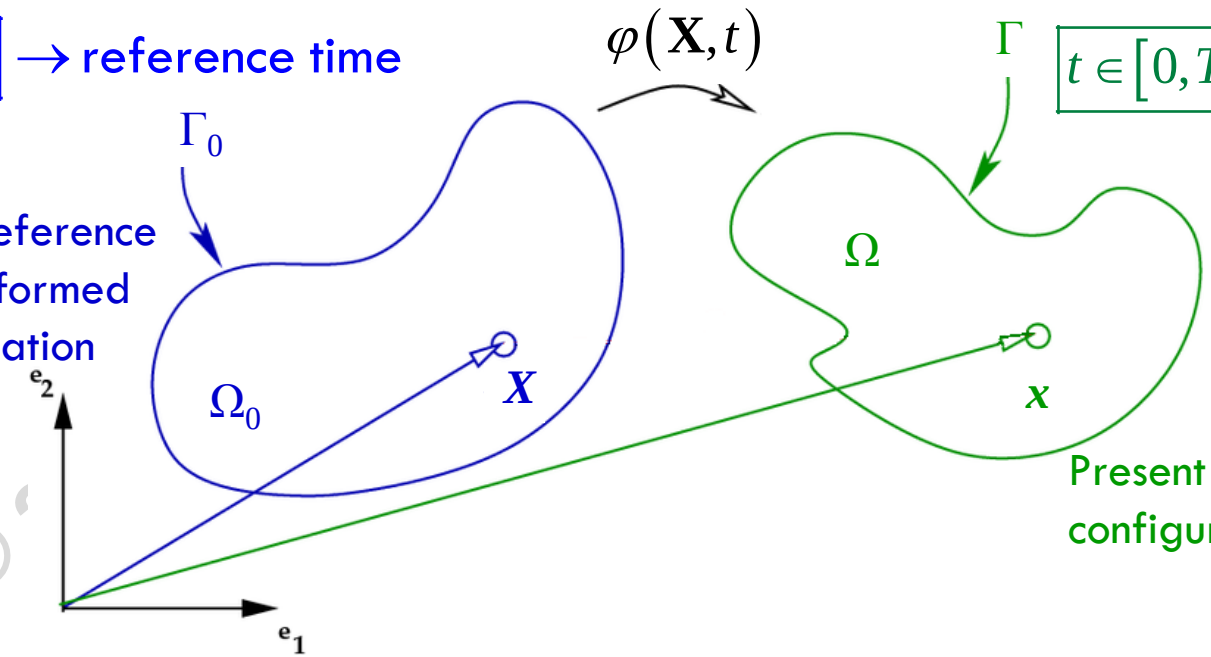
Ω or Ω_t: deformed (or present) configuration, at present time t.

Γ or Γ_t : deformed boundary.

$\boldsymbol{x}$: Position vector of the same particle at present time.

$\boxed{t_0 = 0}$ → reference time

$\boxed{t \in [0, T]}$ → current time



Initial, reference or undeformed configuration

Present or deformed configuration

Material and Spatial Coordinates

- The position vector of a given particle can be expressed in:
  - Non-deformed or Reference Configuration

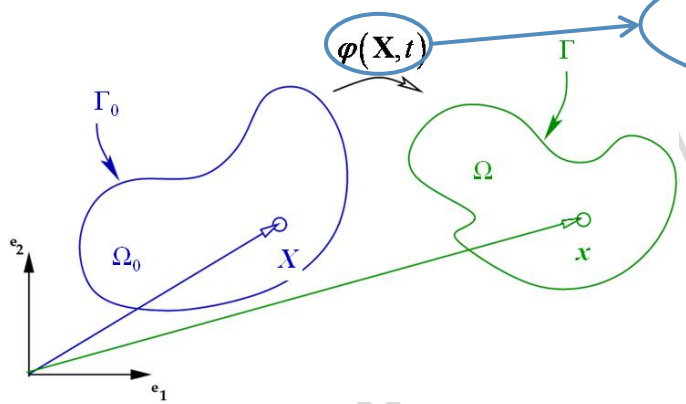
$$[\mathbf{X}] = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \equiv \text{material coordinates (capital letter)}$$

  - Deformed or Present Configuration

$$[\mathbf{x}] = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \equiv \text{spatial coordinates (small letter)}$$

Equations of Motion

- The motion of a given particle is described by the evolution of its spatial coordinates (or its position vector) over time.



$$\begin{cases} \mathbf{x} = \varphi(\text{particle label}, t) = \mathbf{x}(\text{particle label}, t) \\ x_i = \varphi_i(\text{particle label}, t) \qquad i \in \{1,2,3\} \end{cases}$$

φ (particle label, t) is the motion that takes the body from a reference configuration to the current one.

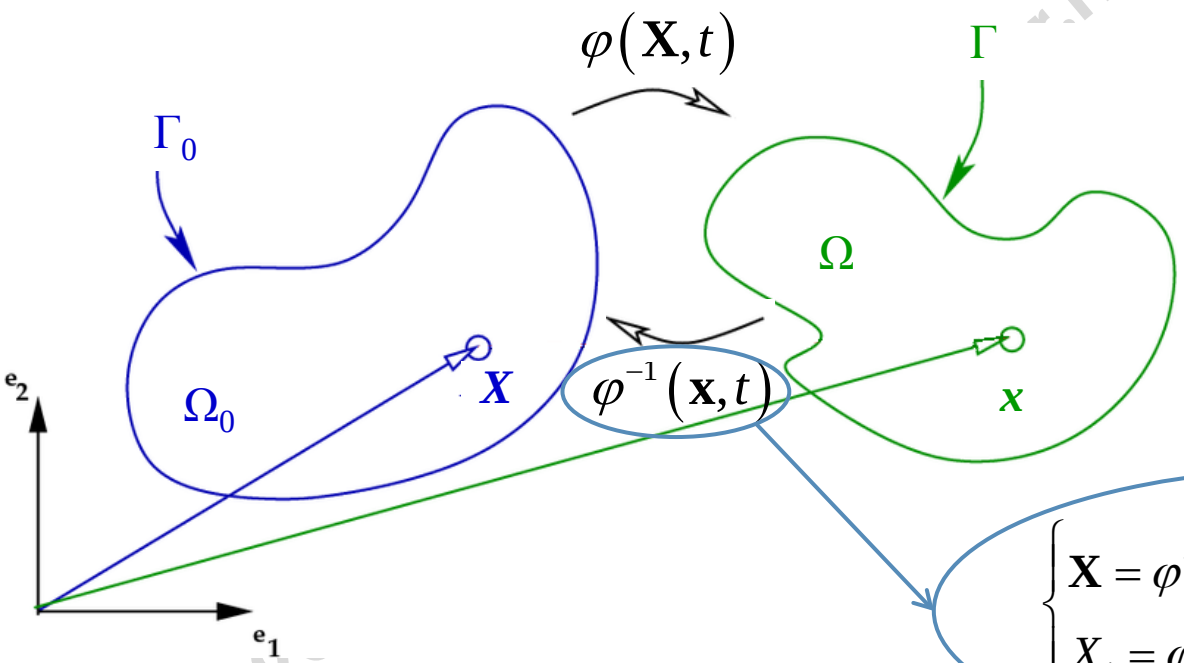
- The **Canonical Form** of the Equations of Motion is obtained when the "particle label" is taken as its material coordinates

$$\text{particle label} \equiv \{X_1 \quad X_2 \quad X_3\}^T \equiv \mathbf{X}$$

$$\begin{cases} \mathbf{x} = \varphi(\mathbf{X},t) \overset{not}{=} \mathbf{x}(\mathbf{X},t) \\ x_i = \varphi_i(X_1, X_2, X_3, t) \quad i \in \{1,2,3\} \end{cases}$$

Inverse Equations of Motion

- The inverse equations of motion give the material coordinates as a function of the spatial ones.



$$\begin{cases} \mathbf{X} = \varphi^{-1}(\mathbf{x},t) \overset{not}{=} \mathbf{X}(\mathbf{x},t) \\ X_i = \varphi_i^{-1}(x_1, x_2, x_3, t) \quad i \in \{1,2,3\} \end{cases}$$

Mathematical restrictions for φ and φ^{-1} defining a "physical" motion

- **Consistency condition**
  - $\varphi(\mathbf{X},0)=\mathbf{X}$, as $\mathbf{X}$ is the position vector for t=0
- **Continuity condition**
  - $\varphi \in C^1$, φ is continuous with continuous derivatives
- **Biunivocity condition**
  - φ is biunivocal to guarantee that two particles do not occupy simultaneously the same spot in space and that a particle does not occupy simultaneously more than one spot in space.
    Mathematically: the "Jacobian" of the motion's equations should be different from zero:

$$J=\left|\frac{\partial\varphi(\mathbf{X},t)}{\partial\mathbf{X}}\right|=\det\left[\frac{\partial\varphi_i}{\partial\mathbf{X}_j}\right]\neq 0$$

- **The "Jacobian" of the equations of motion should be "strictly positive"**

$$J=\left|\frac{\partial\varphi(\mathbf{X},t)}{\partial\mathbf{X}}\right|=\det\left[\frac{\partial\varphi_i}{\partial\mathbf{X}_j}\right]>0$$

⇒ density is always positive (to be proven)

Example

The spatial description of the motion of a continuous medium is given by:

$$\mathbf{x}(\mathbf{X},t) \equiv \begin{cases} x_1 = X_1 e^{2t} \\ x_2 = X_2 e^{-2t} \\ x_3 = 5X_1 t + X_3 e^{2t} \end{cases} \equiv \begin{cases} x = Xe^{2t} \\ y = Ye^{-2t} \\ z = 5Xt + Ze^{2t} \end{cases}$$

Find the inverse equations of motion.

Example - Solution

$$\mathbf{x}(\mathbf{X},t) \equiv \begin{cases} x_1 = X_1 e^{2t} \\ x_2 = X_2 e^{-2t} \\ x_3 = 5X_1 t + X_3 e^{2t} \end{cases}$$

Check the mathematical restrictions:

- **Consistency Condition** $\varphi(\mathbf{X},0) = \mathbf{X}$ **?**

$$\mathbf{x}(\mathbf{X},t=0) = \begin{Bmatrix} X_1 e^{2\cdot 0} \\ X_2 e^{-2\cdot 0} \\ 5X_1 \cdot 0 + X_3 e^{2\cdot 0} \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \mathbf{X}$$

- **Continuity Condition** $\varphi \in C^1$ **?**
- **Biunivocity Condition ?**

$$J = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 5t & 0 & e^{2t} \end{vmatrix} = e^{2t} \cdot e^{-2t} \cdot e^{2t} = e^{2t} \neq 0 \qquad \forall t$$

- **Density positive ?**

$$J = \left| \frac{\partial \varphi(\mathbf{X},t)}{\partial \mathbf{X}} \right| > 0 \qquad J = e^{2t} > 0$$

Example - Solution

$$\mathbf{x}(\mathbf{X},t) \equiv \begin{cases} x_1 = X_1 e^{2t} \\ x_2 = X_2 e^{-2t} \\ x_3 = 5X_1 t + X_3 e^{2t} \end{cases}$$

Calculate the inverse equations:

$$x_1 = X_1 e^{2t} \quad \Rightarrow \quad X_1 = \frac{x_1}{e^{2t}} = x_1 e^{-2t}$$

$$x_2 = X_2 e^{-2t} \quad \Rightarrow \quad X_2 = \frac{x_2}{e^{-2t}} = x_2 e^{2t}$$

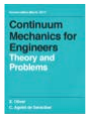
$$x_3 = 5X_1 t + X_3 e^{2t} \quad \Rightarrow \quad X_3 = \frac{x_3 - 5X_1 t}{e^{2t}} = \left(x_3 - 5\left(x_1 e^{-2t}\right)t\right)e^{-2t} = x_3 e^{-2t} - 5t x_1 e^{-4t}$$

$$\boxed{\mathbf{X} \equiv \varphi^{-1}(\mathbf{x},t) = \begin{cases} X_1 = x_1 e^{-2t} \\ X_2 = x_2 e^{2t} \\ X_3 = x_3 e^{-2t} - 5t x_1 e^{-4t} \end{cases}}$$

1.3 Descriptions of Motion

Ch.1. Description of Motion

Link to YouTube video



Descriptions of Motion

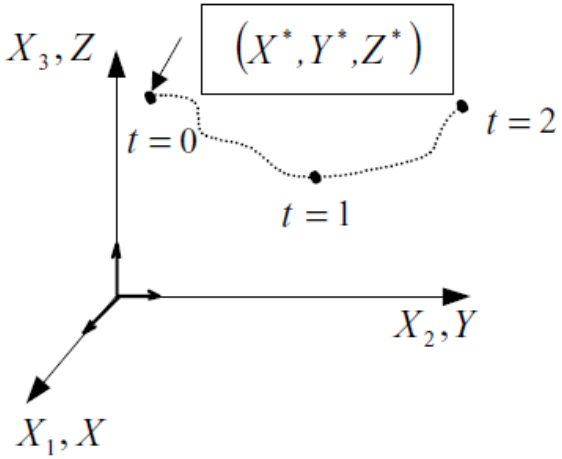
- The mathematical description of the particle properties can be done in two ways:
  - Material (Lagrangian) Description
  - Spatial (Eulerian) Description

Material or Lagrangian Description

- The physical properties are described in terms of the <u>material coordinates</u> and <u>time</u>.

- It focuses on what is occurring at a <u>fixed material point (a</u> particle, labeled by its material coordinates<u>)</u> as time progresses.

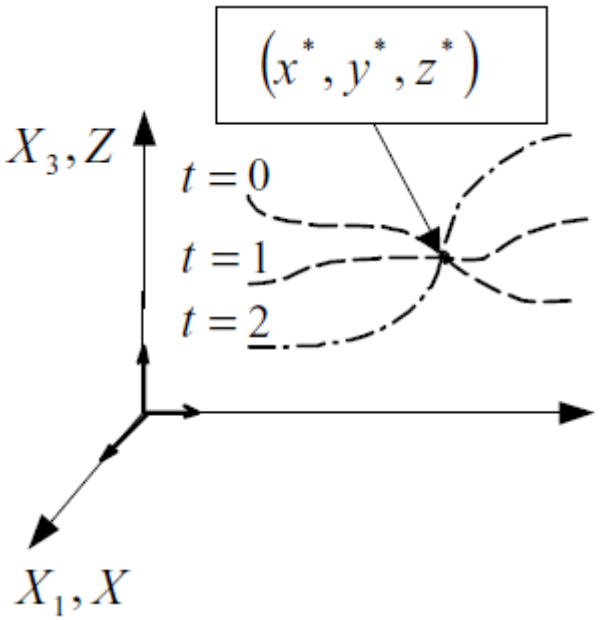
- Normally used in <u>solid mechanics</u>.



Spatial or Eulerian Description

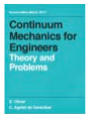
- The physical properties are described in terms of the spatial coordinates and time.

- It focuses on what is occurring at a fixed point in space (a spatial point labeled by its spatial coordinates) as time progresses.



- Normally used in fluid mechanics.

Link to YouTube video



Example

The equation of motion of a continuous medium is:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \equiv \begin{cases} x = X - Yt \\ y = Xt + Y \\ z = -Xt + Z \end{cases}$$

Find the spatial description of the property whose material description is:

$$\bar{\rho}(X, Y, Z, t) = \frac{X + Y + Z}{1 + t^2}$$

Example - Solution

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \equiv \begin{cases} x = X - Yt \\ y = Xt + Y \\ z = -Xt + Z \end{cases}$$

Check the mathematical restrictions:

- **Consistency Condition** $\phi(\mathbf{X}, 0) = \mathbf{X}$ **?**

$$\mathbf{x}(\mathbf{X}, t = 0) = \begin{Bmatrix} X - Y \cdot 0 \\ X \cdot 0 + Y \\ X \cdot 0 + Z \end{Bmatrix} = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \mathbf{X}$$

- **Continuity Condition** $\phi \in C^1$ **?**
- **Biunivocity Condition ?**

$$J = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix} = \begin{vmatrix} 1 & -t & 0 \\ t & 1 & 0 \\ -t & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 - 1 \cdot (-t) \cdot t = 1 + t^2 \neq 0 \quad \forall t$$

- **Any diff. Vol. must be positive** $J = \left| \frac{\partial \phi(\mathbf{X}, t)}{\partial \mathbf{X}} \right| > 0$ **?**

$$J = 1 + t^2 > 0$$

Example - Solution

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \equiv \begin{cases} x = X - Yt \\ y = Xt + Y \\ z = -Xt + Z \end{cases}$$

Calculate the inverse equations:

$$\left.\begin{array}{lll} x = X - Yt & \Rightarrow & X = x + Yt \\ y = Xt + Y & \Rightarrow & X = \dfrac{y - Y}{t} \end{array}\right\} \Rightarrow \quad x + Yt = \frac{y - Y}{t} \quad \Rightarrow \quad Yt^2 + Y = y - xt \quad \Rightarrow \quad Y = \frac{y - xt}{1 + t^2}$$

$$X = x + Yt = x + \left(\frac{y - xt}{1 + t^2}\right)t = \frac{x + xt^2 + yt - xt^2}{1 + t^2} = \frac{x + yt}{1 + t^2}$$

$$z = -Xt + Z \quad \Rightarrow \quad Z = z + Xt = z + \left(\frac{x + yt}{1 + t^2}\right)t = \frac{z + zt^2 + xt + yt^2}{1 + t^2}$$

$$\mathbf{X} \equiv \varphi^{-1}(\mathbf{x}, t) = \begin{cases} X = \dfrac{x + yt}{1 + t^2} \\ Y = \dfrac{y - xt}{1 + t^2} \\ Z = \dfrac{z + zt^2 + xt + yt^2}{1 + t^2} \end{cases}$$

Example - Solution

Calculate the property in its spatial description:

$$\mathbf{X} \equiv \varphi^{-1}(\mathbf{x},t) = \begin{cases} X = \dfrac{x+yt}{1+t^2} \\ Y = \dfrac{y-xt}{1+t^2} \\ Z = \dfrac{z+zt^2+xt+yt^2}{1+t^2} \end{cases}$$

$$\bar{\rho}(X,Y,Z,t) = \frac{X+Y+Z}{1+t^2} = \frac{\left(\dfrac{x+yt}{1+t^2}\right)+\left(\dfrac{y-xt}{1+t^2}\right)+\left(\dfrac{z+zt^2+xt+yt^2}{1+t^2}\right)}{1+t^2} = \frac{x+y+yt+yt^2+z+zt^2}{\left(1+t^2\right)^2}$$

$$\bar{\rho}(X,Y,Z,t) = \frac{X+Y+Z}{1+t^2} \Rightarrow \frac{x+y\left(1+t+t^2\right)+z\left(1+t^2\right)}{\left(1+t^2\right)^2} = \rho\ (x,y,z,t)$$

1.4 Time Derivatives

Ch.1. Description of Motion

Link to YouTube video

Material and Local Derivatives

- The time derivative of a given property can be defined based on the:
  - Material Description $\Gamma(\mathbf{X},t)$ → **TOTAL** or **MATERIAL DERIVATIVE**
    - Variation of the property w.r.t. time following a specific particle in the continuous medium.
    - $$\text{material derivative} \equiv \frac{\partial \Gamma(\mathbf{X},t)}{\partial t} \rightarrow \begin{cases} \text{partial time derivative of the} \\ \textbf{material description} \text{ of the propery} \end{cases}$$
  - Spatial Description $\gamma(\mathbf{x},t)$ → **LOCAL** or **SPATIAL DERIVATIVE**
    - Variation of the property w.r.t. time in a fixed spot of space.
    - $$\text{local derivative} \equiv \frac{\partial \gamma(\mathbf{x},t)}{\partial t} \rightarrow \begin{cases} \text{partial time derivative of the} \\ \textbf{spatial description} \text{ of the propery} \end{cases}$$

Convective Derivative

- Remember: $\mathbf{x}=\mathbf{x}(\mathbf{X},t)$, therefore, $\gamma(\mathbf{x},t)=\gamma(\mathbf{x}(\mathbf{X},t),t)=\Gamma(\mathbf{X},t)$
- The material derivative can be computed in terms of spatial descriptions:

$$\text{material derivative} \rightarrow \overset{not}{=} \frac{d}{dt}\gamma(\mathbf{x},t) \overset{not}{=} \frac{D}{Dt}\gamma(\mathbf{x},t) = \frac{\partial\Gamma(\mathbf{X},t)}{\partial t} =$$

$$= \frac{d}{dt}\gamma(\mathbf{x}(\mathbf{X},t),t) = \frac{\partial\gamma(\mathbf{x},t)}{\partial t} + \underbrace{\frac{\partial\gamma}{\partial x_i}}_{[\nabla\gamma]_i} \cdot \underbrace{\frac{\partial x_i}{\partial t}}_{\left[\frac{\partial\mathbf{x}}{\partial t}\right]_i} = \frac{\partial\gamma(\mathbf{x},t)}{\partial t} + \underbrace{\underbrace{\frac{\partial\gamma}{\partial\mathbf{x}}}_{\nabla\gamma(\mathbf{x},t)} \cdot \underbrace{\frac{\partial\mathbf{x}}{\partial t}}_{\mathbf{v}(\mathbf{x},t)}}_{\mathbf{v}(\mathbf{x},t)\cdot\nabla\gamma(\mathbf{x},t)} =$$

- Generalising for any property:

$$\frac{d\chi(\mathbf{x},t)}{dt} = \frac{\partial\chi(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla\chi(\mathbf{x},t)$$

material derivative

local derivative

convective rate of change (convective derivative)

REMARK

The spatial Nabla operator is defined as: $\nabla \equiv \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i$

Convective Derivative

- Convective rate of change or convective derivative is implicitly defined as:

$$\mathbf{v}\cdot\nabla(\bullet)$$

- The term *convection* is generally applied to <u>motion</u> related phenomena.
  - If there is no convection ($\mathbf{v}=0$) there is no convective rate of change and the material and local derivatives coincide.

$$\mathbf{v}\cdot\nabla(\bullet)=0 \quad \Rightarrow \quad \frac{d(\bullet)}{dt}=\frac{\partial(\bullet)}{\partial t}$$

Example

Given the following equation of motion:

$$\mathbf{x}(\mathbf{X},t) \equiv \begin{cases} x = X + Yt + Zt \\ y = Y + 2Zt \\ z = Z + 3Xt \end{cases}$$

And the spatial description of a property $\rho(\mathbf{x},t) = 3x + 2y + 3t$,

Calculate its material derivative.

Option #1: Computing the material derivative from material descriptions

Option #2: Computing the material derivative from spatial descriptions

Example - Solution

$$\mathbf{x}(\mathbf{X},t) \equiv \begin{cases} x = X + Yt + Zt \\ y = Y + 2Zt \\ z = Z + 3Xt \end{cases}$$

$$\rho(\mathbf{x},t) = 3x + 2y + 3t$$

Option #1: Computing the material derivative from material descriptions

Obtain ρ as a function of $\mathbf{X}$ by replacing the Eqns. of motion into $\rho(\mathbf{x},t)$:

$$\rho(\mathbf{x},t) = \rho(\mathbf{x}(\mathbf{X},t),t) = \bar{\rho}(\mathbf{X},t) = 3(X + Yt + Zt) + 2(Y + 2Zt) + 3t$$
$$= 3X + 3Yt + 2Y + 7Zt + 3t$$

Calculate its material derivative as the partial derivative of the material description:

$$\left.\frac{d\rho(\mathbf{x},t)}{dt}\right|_{\mathbf{x}=\mathbf{x}(\mathbf{X},t)} = \frac{\partial\bar{\rho}(\mathbf{X},t)}{\partial t} = \frac{\partial}{\partial t}(3X + 3Yt + 2Y + 7Zt + 3t) = 3Y + 7Z + 3$$

$$\boxed{\left.\frac{d\rho(\mathbf{x},t)}{dt}\right|_{\mathbf{x}=\mathbf{x}(\mathbf{X},t)} = \frac{\partial\bar{\rho}(\mathbf{X},t)}{\partial t} = 3 + 3Y + 7Z}$$

Example - Solution

$$\mathbf{x}(\mathbf{X},t) \equiv \begin{cases} x = X + Yt + Zt \\ y = Y + 2Zt \\ z = Z + 3Xt \end{cases}$$

$$\rho(\mathbf{x},t) = 3x + 2y + 3t$$

Option #2: Computing the material derivative from spatial descriptions

$$\frac{d\rho(\mathbf{x},t)}{dt} = \frac{\partial \rho(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla\rho(\mathbf{x},t)$$

Applying this on $\rho(\mathbf{x},t)$:

$$\begin{cases} \dfrac{\partial \rho(\mathbf{x},t)}{\partial t} = \dfrac{\partial}{\partial t}(3x+2y+3t) = 3 \\[2ex] \mathbf{v}(\mathbf{x},t)\big|_{\mathbf{x}=\mathbf{x}(\mathbf{X},t)} = \dfrac{\partial \mathbf{x}}{\partial t} = \left[\dfrac{\partial}{\partial t}(X+Yt+Zt),\ \dfrac{\partial}{\partial t}(Y+2Zt),\ \dfrac{\partial}{\partial t}(Z+3Xt)\right]^T = [Y+Z, 2Z\ ,\ 3X]^T = \begin{bmatrix} Y+Z \\ 2Z \\ 3X \end{bmatrix} \\[2ex] \nabla\rho(\mathbf{x},t) = \left[\dfrac{\partial \rho(\mathbf{x},t)}{\partial x},\ \dfrac{\partial \rho(\mathbf{x},t)}{\partial y},\ \dfrac{\partial \rho(\mathbf{x},t)}{\partial z}\right]^T = \left[\dfrac{\partial}{\partial x}(3x+2y+3t),\ \dfrac{\partial}{\partial y}(3x+2y+3t),\ \dfrac{\partial}{\partial z}(3x+2y+3t)\right]^T = \\[2ex] \qquad = [3,\ 2,\ 0]^T = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \end{cases}$$

Example - Solution

$$\mathbf{x}(\mathbf{X},t) \equiv \begin{cases} x = X + Yt + Zt \\ y = Y + 2Zt \\ z = Z + 3Xt \end{cases}$$

$$\rho(\mathbf{x},t) = 3x + 2y + 3t$$

Option #2

The material derivative is obtained:

$$\left.\frac{d\rho(\mathbf{x},t)}{dt}\right|_{\mathbf{x}=\mathbf{x}(\mathbf{X},t)} = 3 + \underbrace{\underbrace{[Y+Z,\ 2Z,\ 3X]}_{[\mathbf{v}]^T}\underbrace{\begin{bmatrix}3\\2\\0\end{bmatrix}}_{[\nabla\rho]}}_{\mathbf{v}\cdot(\nabla\rho)} = 3 + 3Y + 3Z + 4Z$$

$$\boxed{\left.\frac{d\rho(\mathbf{x},t)}{dt}\right|_{\mathbf{x}=\mathbf{x}(\mathbf{X},t)} = 3 + 3Y + 7Z}$$

1.5 Velocity and Acceleration

Ch.1. Description of Motion

Velocity

- **Time derivative** of the equations of motion.
  - Material description of the velocity:

    Time derivative of the equations of motion

$$\begin{cases} \mathbf{V}(\mathbf{X},t) = \dfrac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} \\ V_i(\mathbf{X},t) = \dfrac{\partial x_i(\mathbf{X},t)}{\partial t} \quad i \in \{1,2,3\} \end{cases}$$

REMARK

Remember the equations of motion are of the form:

$$\mathbf{x} = \varphi(\mathbf{X},t) \overset{not}{=} \mathbf{x}(\mathbf{X},t)$$

  - Spatial description of the velocity:

    Velocity is expressed in terms of $\mathbf{x}$ using the inverse equations of motion:

$$\mathbf{V}(\mathbf{X}(\mathbf{x},t),t) \Rightarrow \mathbf{v}(\mathbf{x},t)$$

Acceleration

- **Material time derivative** of the velocity field.
  - Material description of acceleration:

    Derivative of the material description of velocity:

$$\begin{cases} \mathbf{A}(\mathbf{X},t) = \dfrac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t} \\ A_i(\mathbf{X},t) = \dfrac{\partial V_i(\mathbf{X},t)}{\partial t} \quad i \in \{1,2,3\} \end{cases}$$

  - Spatial description of acceleration:

    $\mathbf{A}(\mathbf{X},t)$ is expressed in terms of $\mathbf{x}$ using the inverse equations of motion:

$$\mathbf{A}(\mathbf{X}(\mathbf{x},t),t) \Rightarrow \mathbf{a}(\mathbf{x},t)$$

    Or $\mathbf{a}(\mathbf{x},t)$ is obtained directly through the material derivative of $\mathbf{v}(\mathbf{x},t)$:

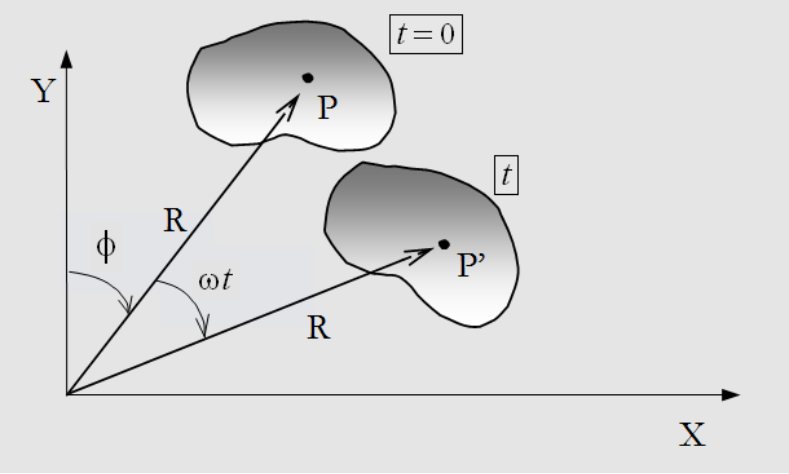
$$\begin{cases} \mathbf{a}(\mathbf{x},t) = \dfrac{d\mathbf{v}(\mathbf{x},t)}{dt} = \dfrac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla\mathbf{v}(\mathbf{x},t) \\ a_i(\mathbf{x},t) = \dfrac{d\mathrm{v}_i(\mathbf{x},t)}{dt} = \dfrac{\partial \mathrm{v}_i(\mathbf{x},t)}{\partial t} + \mathrm{v}_k(\mathbf{x},t)\cdot\dfrac{\partial \mathrm{v}_i}{\partial x_k}(\mathbf{x},t) \quad i \in \{1,2,3\} \end{cases}$$

Example

Consider a solid that rotates at a constant angular velocity ω and has the following equation of motion:

$$\mathbf{x}\,(\underbrace{R,\ \phi}_{\text{label of particle}},\ t) \rightarrow \begin{cases} x = R\ \sin(\omega t + \phi) \\ y = R\ \cos(\omega t + \phi) \end{cases}$$

$\rightarrow$ (non - canonical equations of motion)



Find the velocity and acceleration of the movement described in both, material and spatial forms.

Example - Solution

$$\begin{cases} x = R\ \sin(\omega t + \phi) \\ y = R\ \cos(\omega t + \phi) \end{cases}$$

Using the expressions $\begin{cases} \sin(a \pm b) = \sin a \cdot \cos b \pm \cos a \cdot \sin b \\ \cos(a \pm b) = \cos a \cdot \cos b \mp \sin a \cdot \sin b \end{cases}$

The equation of motion can be rewritten as:

$$\begin{cases} x = R\ \sin(\omega t + \phi) = R\ \sin(\omega t)\cos\phi + R\ \cos(\omega t)\ \sin\phi \\ y = R\ \cos(\omega t + \phi) = R\ \cos(\omega t)\cos\phi - R\ \sin(\omega t)\ \sin\phi \end{cases}$$

For t=0, the equation of motion becomes: $\begin{cases} X = R\ \sin\phi \\ Y = R\ \cos\phi \end{cases}$

Therefore, the equation of motion in terms of the material coordinates is:

$$x = R\ \sin(\omega t + \phi) = \overbrace{R\ \sin(\omega t)\cos\phi}^{=Y} + \overbrace{R\ \cos(\omega t)\ \sin\phi}^{=X} = X\cos(\omega t) + Y\sin(\omega t)$$

$$y = R\ \cos(\omega t + \phi) = \underbrace{R\ \cos(\omega t)\cos\phi}_{=Y} - \underbrace{R\ \sin(\omega t)\ \sin\phi}_{=X} = -X\sin(\omega t) + Y\cos(\omega t)$$

Example - Solution

$$x = X\cos(\omega t) + Y\sin(\omega t)$$
$$y = -X\sin(\omega t) + Y\cos(\omega t)$$

The inverse equation of motion is easily obtained

$$x = X\cos(\omega t) + Y\sin(\omega t) \Rightarrow X = \frac{x - Y\sin(\omega t)}{\cos(\omega t)}$$

$$y = -X\sin(\omega t) + Y\cos(\omega t) \Rightarrow X = \frac{-y + Y\cos(\omega t)}{\sin(\omega t)}$$

$$\frac{x - Y\sin(\omega t)}{\cos(\omega t)} = \frac{-y + Y\cos(\omega t)}{\sin(\omega t)}$$

$$x\sin(\omega t) - Y\sin^2(\omega t) = -y\cos(\omega t) + Y\cos^2(\omega t)$$

$$x\sin(\omega t) + y\cos(\omega t) = Y\underbrace{\left(\cos^2(\omega t) + \sin^2(\omega t)\right)}_{=1}$$

$$X = \frac{x - \left(x\sin(\omega t) + y\cos(\omega t)\right)\sin(\omega t)}{\cos(\omega t)} = \frac{x}{\cos(\omega t)} - \frac{x\sin^2(\omega t)}{\cos(\omega t)} - \frac{y\cos(\omega t)\sin(\omega t)}{\cos(\omega t)} =$$

$$= x\underbrace{\frac{1 - \sin^2(\omega t)}{\cos(\omega t)}}_{=\cos(\omega t)} - y\sin(\omega t)$$

Example - Solution

So, the equation of motion and its inverse in terms of the material coordinates are:

$$\mathbf{x}(\mathbf{X},t) \rightarrow \begin{cases} x = X\cos(\omega t) + Y\sin(\omega t) \\ y = -X\sin(\omega t) + Y\cos(\omega t) \end{cases} \rightarrow \text{canonical equations of motion}$$

$$\mathbf{X}(\mathbf{x},t) \rightarrow \begin{cases} X = x\cos(\omega t) - y\sin(\omega t) \\ Y = x\sin(\omega t) + y\cos(\omega t) \end{cases} \rightarrow \text{inverse equations of motion}$$

Example - Solution

$$\mathbf{x}(\mathbf{X},t) \rightarrow \begin{cases} x = X\cos(\omega t) + Y\sin(\omega t) \\ y = -X\sin(\omega t) + Y\cos(\omega t) \end{cases}$$

Velocity in material description is obtained from $\mathbf{V}(\mathbf{X},t) = \dfrac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}$

$$\mathbf{V}(\mathbf{X},t) = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \begin{cases} \dfrac{\partial x}{\partial t} = \dfrac{\partial}{\partial t}\left(X\cos(\omega t) + Y\sin(\omega t)\right) \\ \dfrac{\partial y}{\partial t} = \dfrac{\partial}{\partial t}\left(-X\sin(\omega t) + Y\cos(\omega t)\right) \end{cases}$$

$$\boxed{\mathbf{V}(\mathbf{X},t) = \begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \begin{Bmatrix} -X\ \omega\ \sin(\omega t) + Y\ \omega\ \cos(\omega t) \\ -X\ \omega\ \cos(\omega t) - Y\ \omega\ \sin(\omega t) \end{Bmatrix}}$$

Example - Solution

$$\begin{cases} X = x\cos(\omega t) - y\sin(\omega t) \\ Y = x\sin(\omega t) + y\cos(\omega t) \end{cases}$$

$$\begin{cases} x = X\cos(\omega t) + Y\sin(\omega t) \\ y = -X\sin(\omega t) + Y\cos(\omega t) \end{cases}$$

Velocity in spatial description is obtained introducing $\mathbf{X}(\mathbf{x},t)$ into $\mathbf{V}(\mathbf{X},t)$:

$$\boxed{\begin{cases} x = X\cos(\omega t) + Y\sin(\omega t) \\ y = -X\sin(\omega t) + Y\cos(\omega t) \end{cases}} \rightarrow \mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X}(\mathbf{x},t),t) = \begin{Bmatrix} \mathrm{v}_x \\ \mathrm{v}_y \end{Bmatrix} = \begin{Bmatrix} \overbrace{\left[-X\sin(\omega t) + Y\cos(\omega t)\right]}^{y}\omega \\ \underbrace{\left[-X\cos(\omega t) - Y\sin(\omega t)\right]}_{-x}\omega \end{Bmatrix} = \begin{Bmatrix} \omega y \\ -\omega x \end{Bmatrix}$$

Alternative procedure (longer):

$$\mathbf{v}(\mathbf{x},t) = \begin{Bmatrix} -\omega x\cos(\omega t)\sin(\omega t) + \omega y\sin^2(\omega t) + \omega x\sin(\omega t)\cos(\omega t) + \omega y\cos^2(\omega t) \\ -\omega x\cos^2(\omega t) + \omega y\sin(\omega t)\cos(\omega t) - \omega x\sin^2(\omega t) - \omega y\cos(\omega t)\sin(\omega t) \end{Bmatrix} =$$

$$= \begin{Bmatrix} \omega x\underbrace{\left(\sin(\omega t)\cos(\omega t) - \cos(\omega t)\sin(\omega t)\right)}_{=0} + \omega y\underbrace{\left(\sin^2(\omega t) + \cos^2(\omega t)\right)}_{=1} \\ -\omega x\underbrace{\left(\sin^2(\omega t) + \cos^2(\omega t)\right)}_{=1} + \omega y\underbrace{\left(\sin(\omega t)\cos(\omega t) - \cos(\omega t)\sin(\omega t)\right)}_{=0} \end{Bmatrix}$$

$$\boxed{\mathbf{v}(\mathbf{x},t) = \begin{Bmatrix} \mathrm{v}_x \\ \mathrm{v}_y \end{Bmatrix} = \begin{Bmatrix} \omega\, y \\ -\omega\, x \end{Bmatrix}}$$

Example - Solution

$$\mathbf{V}(\mathbf{X},t)=\begin{Bmatrix}-X\ \omega\ \sin(\omega t)+Y\ \omega\ \cos(\omega t)\\ -X\ \omega\ \cos(\omega t)-Y\ \omega\ \sin(\omega t)\end{Bmatrix}$$

Acceleration in material description is obtained applying: $\mathbf{A}(\mathbf{X},t)=\dfrac{\partial\mathbf{V}(\mathbf{X},t)}{\partial t}$

$$\mathbf{A}(\mathbf{X},t)=\frac{\partial\mathbf{V}(\mathbf{X},t)}{\partial t}=\begin{cases}\dfrac{\partial V_x}{\partial t}=-X\ \omega^2\ \cos(\omega t)-Y\ \omega^2\ \sin(\omega t)\\ \dfrac{\partial V_y}{\partial t}=X\ \omega^2\ \sin(\omega t)-Y\ \omega^2\ \cos(\omega t)\end{cases}$$

$$\boxed{\mathbf{A}(\mathbf{X},t)=\begin{Bmatrix}A_x\\ A_y\end{Bmatrix}=-\omega^2\begin{Bmatrix}X\cos(\omega t)+Y\sin(\omega t)\\ -X\sin(\omega t)+Y\cos(\omega t)\end{Bmatrix}}$$

Example - Solution

$$\begin{cases} X = x\cos(\omega t) - y\sin(\omega t) \\ Y = x\sin(\omega t) + y\cos(\omega t) \end{cases}$$

(**OPTION #1**) **Acceleration in spatial description** is obtained by replacing the inverse equation of motion into $\mathbf{A}(\mathbf{X},t)$:

$$\mathbf{a}(\mathbf{x},t) = \mathbf{A}(\mathbf{X}(\mathbf{x},t),t) =$$

$$= -\omega^2 \begin{Bmatrix} (x\cos(\omega t) - y\sin(\omega t))\cos(\omega t) + (x\sin(\omega t) + y\cos(\omega t))\sin(\omega t) \\ -(x\cos(\omega t) - y\sin(\omega t))\sin(\omega t) + (x\sin(\omega t) + y\cos(\omega t))\cos(\omega t) \end{Bmatrix}$$

$$\mathbf{a}(\mathbf{x},t) = -\omega^2 \begin{Bmatrix} x\underbrace{(\cos^2(\omega t) + \sin^2(\omega t))}_{=1} + y\underbrace{(-\sin(\omega t)\cos(\omega t) + \cos(\omega t)\sin(\omega t))}_{=0} \\ x\underbrace{(-\cos(\omega t)\sin(\omega t) + \sin(\omega t)\cos(\omega t))}_{=0} + y\underbrace{(\sin^2(\omega t) + \cos^2(\omega t))}_{=1} \end{Bmatrix}$$

$$\boxed{\mathbf{a}(\mathbf{x},t) = \begin{Bmatrix} a_x \\ a_y \end{Bmatrix} = \begin{Bmatrix} -\omega^2 x \\ -\omega^2 y \end{Bmatrix}}$$

Example - Solution

$$\mathbf{v}(\mathbf{x},t)=\begin{Bmatrix} v_x \\ v_y \end{Bmatrix}=\begin{Bmatrix} \omega\, y \\ -\omega\, x \end{Bmatrix}$$

(**OPTION #2**): **Acceleration in spatial description** is obtained by directly calculating the material derivative of the velocity in spatial description:

$$\mathbf{a}(\mathbf{x},t)=\frac{d\mathbf{v}(\mathbf{x},t)}{dt}=\frac{\partial\mathbf{v}(\mathbf{x},t)}{\partial t}+\mathbf{v}(\mathbf{x},t)\cdot\nabla\mathbf{v}(\mathbf{x},t)$$

$$\mathbf{a}(\mathbf{x},t)=\frac{\partial}{\partial t}\begin{Bmatrix} \omega y \\ -\omega x \end{Bmatrix}+\begin{bmatrix} \omega y & , & -\omega x \end{bmatrix}\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix}\begin{bmatrix} \omega y & , & -\omega x \end{bmatrix}=$$

$$=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}+\begin{bmatrix} \omega y & , & -\omega x \end{bmatrix}\begin{bmatrix} \dfrac{\partial}{\partial x}(\omega y) & \dfrac{\partial}{\partial x}(-\omega x) \\ \dfrac{\partial}{\partial y}(\omega y) & \dfrac{\partial}{\partial y}(-\omega x) \end{bmatrix}=\begin{bmatrix} \omega y & , & -\omega x \end{bmatrix}\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}=\begin{Bmatrix} -\omega^2 x \\ -\omega^2 y \end{Bmatrix}$$

$$\mathbf{a}(\mathbf{x},t)=\begin{Bmatrix} -\omega^2 x \\ -\omega^2 y \end{Bmatrix}$$

1.6 Stationarity and Uniformity

Ch.1. Description of Motion

Link to YouTube video

Stationary properties

- A property is **stationary** when its <u>spatial description</u> is not dependent on time.

$$\chi(\mathbf{x},t)=\chi(\mathbf{x})$$

  - The local derivative of a stationary property is zero.

$$\chi(\mathbf{x},t)=\chi(\mathbf{x}) \quad \Leftrightarrow \quad \frac{\partial\chi(\mathbf{x},t)}{\partial t}=0$$

  - The time-independence in the spatial description (stationarity) does not imply time-independence in the material description:

$$\chi(\mathbf{x},t)=\chi(\mathbf{x}) \quad \not\Leftrightarrow \quad \bar{\chi}(\mathbf{X},t)=\bar{\chi}(\mathbf{X})$$

REMARK
In certain fields, the term **steady-state** is more commonly used.

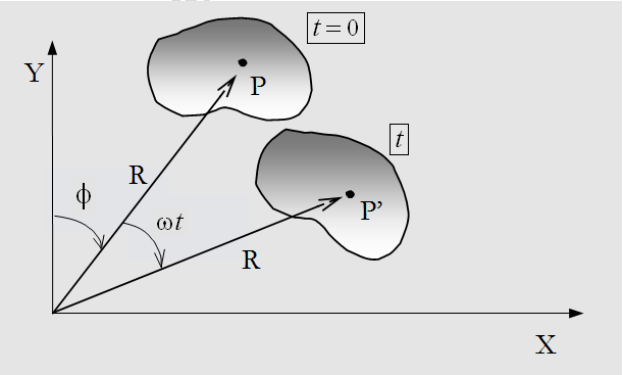
REMARK
This is easily understood if we consider, for example, a stationary velocity:

$$\mathbf{v}(\mathbf{x},t)=\mathbf{v}(\mathbf{x})=\mathbf{v}(\mathbf{x}(\mathbf{X},t))=\mathbf{V}(\mathbf{X},t)$$

Example

Consider a solid that rotates at a constant angular velocity ω and has the following equation of motion:

$$\begin{cases} x = R \quad \sin(\omega t + \varphi) \\ y = R \quad \cos(\omega t + \varphi) \end{cases}$$



We have obtained:

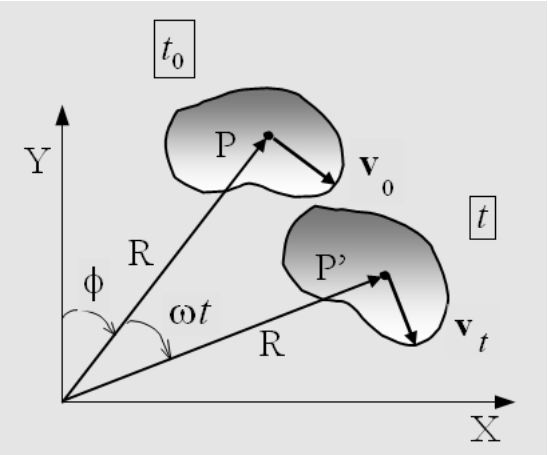
Velocity in spatial description

$$\mathbf{v}(\mathbf{x},t) = \begin{Bmatrix} \mathrm{v}_x \\ \mathrm{v}_y \end{Bmatrix} = \begin{Bmatrix} \omega\, y \\ -\omega\, x \end{Bmatrix}$$

stationary

Velocity in material description

$$\mathbf{V}(\mathbf{X},t) = \begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \begin{Bmatrix} -X\ \omega\ \sin(\omega t) + Y\ \omega\ \cos(\omega t) \\ -X\ \omega\ \cos(\omega t) - Y\ \omega\ \sin(\omega t) \end{Bmatrix}$$



Uniform properties

- A property is **uniform** when its <u>spatial description</u> is not dependent on the spatial coordinates.

$$\chi(\mathbf{x},t) = \chi(t)$$

  - If its spatial description does not depend on the coordinates (uniform character of the property), neither does its material one.

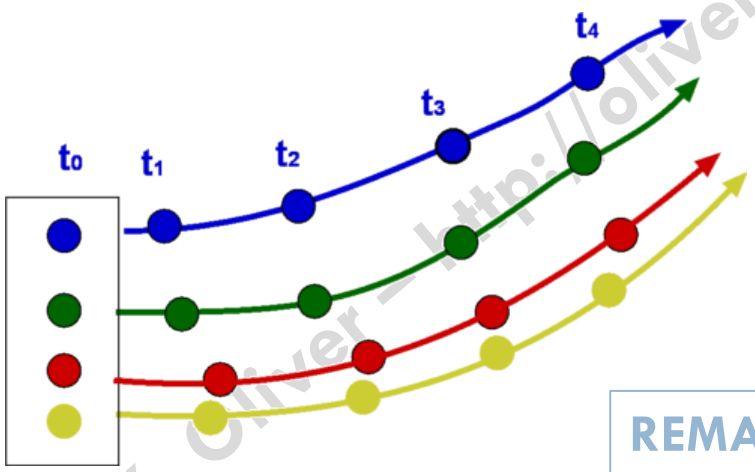
$$\chi(\mathbf{x},t) = \chi(t) \quad \Leftrightarrow \quad \bar{\chi}(\mathbf{X},t) = \chi(t)$$

1.7 Trajectory (path-line)

Ch.1. Description of Motion

Trajectory or pathline

- A **trajectory** or **pathline** is the locus of the <u>positions occupied by a given particle</u> in space throughout time.



REMARK

A trajectory can also be defined as the path that a particle follows through space as a function of time.

Equation of the trajectories

- The equation of a given particle's trajectory is obtained <u>particularizing the equation of motion</u> for that particle, which is identified by it material coordinates $\mathbf{X}^*$.

$$\begin{cases} \mathbf{x}(t) = \boldsymbol{\varphi}(\mathbf{X},t)\big|_{\mathbf{X}=\mathbf{X}^*} = \boldsymbol{\phi}(t) \\ x_i(t) = \varphi_i(\mathbf{X},t)\big|_{\mathbf{X}=\mathbf{X}^*} = \phi_i(t) \quad i \in \{1,2,3\} \end{cases}$$

- Also, from the velocity field in spatial description, $\mathbf{v}(\mathbf{x},t)$:
  - A family of curves is obtained from:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t),t) \Rightarrow \mathbf{x} = \boldsymbol{\phi}(C_1,C_2,C_3,t) \qquad [1]$$

  - Particularizing for a given particle by imposing the consistency condition in the reference configuration:

$$\mathbf{x}(t)\big|_{t=0} = \mathbf{X} \Rightarrow \mathbf{X} = \boldsymbol{\phi}(C_1,C_2,C_3,0) \Rightarrow C_i = \chi_i(\mathbf{X}) \qquad [2]$$

  - Replacing [2] in [1], the equation of the trajectories in canonical form

$$\mathbf{x} = \phi(C_1(\mathbf{X}),C_2(\mathbf{X}),C_3(\mathbf{X}),t) = \varphi(\mathbf{X},t)$$

Example

Consider the following velocity field:

$$\mathbf{v}(\mathbf{x},t)=\begin{Bmatrix}\omega\, y\\ -\omega\, x\end{Bmatrix}$$

Obtain the equation of the trajectories.

Example - Solution

We integrate the velocity field:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x},t) \quad \Rightarrow \quad \begin{cases} \dfrac{dx(t)}{dt} = \mathrm{v}_x(\mathbf{x},t) = \omega y \\ \dfrac{dy(t)}{dt} = \mathrm{v}_y(\mathbf{x},t) = -\omega x \end{cases}$$

This is a crossed-variable system of differential equations. We derive the 2nd eq. and replace it in the 1st one,

$$\frac{d^2y(t)}{dt^2} = -\omega\frac{dx(t)}{dt} = -\omega^2 y(t) \quad \Rightarrow \quad y'' + \omega^2 y = 0$$

Example - Solution

The characteristic equation: $r^2 + \omega^2 = 0$

Has the characteristic solutions: $r_j = \pm\, i\ \omega \quad j \in \{1, 2\}$

And the solution of the problem is:

$$y(t) = Real\ Part\left\{Z_1 e^{iwt} + Z_2 e^{-iwt}\right\} = C_1 \cos(\omega t) + C_2\ \sin(\omega t)$$

And, using $\dfrac{dy}{dt} = -\omega x$, we obtain

$$x = -\frac{1}{\omega}\frac{dy}{dt} = -\frac{1}{\omega}\left(-C_1 \omega \sin(\omega t) + C_2\ \omega \cos(\omega t)\right)$$

So, the general solution is:

$$\begin{cases} x(C_1, C_2, t) = C_1\ \sin(\omega t) - C_2\ \cos(\omega t) \\ y(C_1, C_2, t) = C_1\ \cos(\omega t) + C_2\ \sin(\omega t) \end{cases}$$

Example - Solution

$$\begin{cases} x(C_1,C_2,t) = C_1 \ \sin(\omega t) - C_2 \ \cos(\omega t) \\ y(C_1,C_2,t) = C_1 \ \cos(\omega t) + C_2 \ \sin(\omega t) \end{cases}$$

The canonical form is obtained from the initial conditions:

$$\mathbf{x}(C_1,C_2,0) = \mathbf{X} \quad \Rightarrow \quad \begin{cases} X = x(C_1,C_2,0) = C_1 \ \overbrace{\sin(\omega \cdot 0)}^{=0} - C_2 \ \overbrace{\cos(\omega \cdot 0)}^{=1} = C_2 \\ Y = y(C_1,C_2,0) = C_1 \ \underbrace{\cos(\omega \cdot 0)}_{=1} + C_2 \ \underbrace{\sin(\omega \cdot 0) = C_1}_{=0} \end{cases}$$

This results in:

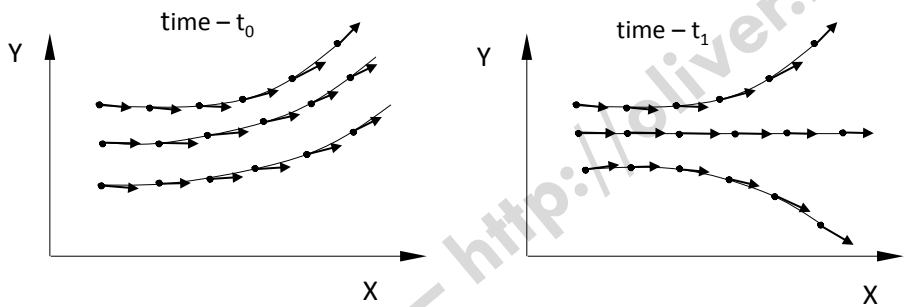
$$\boxed{\mathbf{x}(\mathbf{X},t) \ \rightarrow \ \begin{cases} x = Y \ \sin(\omega t) + X \ \cos(\omega t) \\ y = Y \ \cos(\omega t) - X \ \sin(\omega t) \end{cases}}$$

1.8 Streamline

Ch.1. Description of Motion

Streamline

- The **streamlines** are a family of curves which, for every instant in time, are the <u>velocity field envelopes</u>.



REMARK
Two streamlines can never cut each other. Is it true?

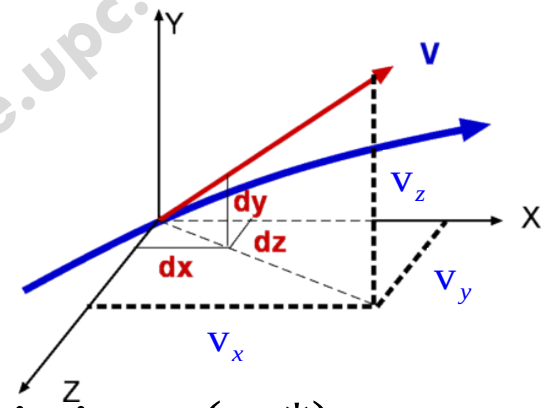
- Streamlines are defined for any given time instant and change with the velocity field.

REMARK
The **envelopes** of vector field are the curves whose tangent vector at each point coincides (in direction and sense but not necessarily in magnitude) with the corresponding vector of the vector field.

Equation of the Streamlines

- The equation of the streamlines is of the type:

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} = d\lambda (= ds) \quad \Rightarrow \quad \frac{d\mathbf{x}}{d\lambda} = \mathbf{v}$$



- Also, from the velocity field in spatial description, $\mathbf{v}(\mathbf{x},t^*)$ at a given time instant t^*:
  - A family of curves is obtained from:

$$\frac{d\mathbf{x}(\lambda)}{d\lambda} = \mathbf{v}\left(\mathbf{x}(\lambda), t^*\right) \quad \Rightarrow \quad \mathbf{x} = \phi\left(C_1', C_2', C_3', \lambda, t^*\right)$$

    - Where each group (C_1', C_2', C_3') identifies a streamline $\mathbf{x}(\lambda)$ whose points are obtained assigning values to the parameter λ.
    - For each time instant t^* a new family of curves is obtained.

Trajectories and Streamlines

- For a <u>stationary</u> velocity field, the <u>trajectories and the streamlines coincide</u> – PROOF:
  1. If $\mathbf{v}(\mathbf{x},t)=\mathbf{v}(\mathbf{x})$:
     - Eq. trajectories:

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{v}\left(\mathbf{x}(t),\cancel{t}\right) \Rightarrow \mathbf{x}=\phi\left(C_1,C_2,C_3,t\right)$$

     - Eq. streamlines:

$$\frac{d\mathbf{x}(\lambda)}{d\lambda}=\mathbf{v}\left(\mathbf{x}(\lambda),\cancel{t^*}\right) \Rightarrow \mathbf{x}=\phi\left(C_1,C_2,C_3,\lambda,\cancel{t^*}\right)$$

The differential equations only differ in the denomination of the integration parameter (t or λ), so the solution to both systems MUST be the same.

Trajectories and Streamlines

- For a <u>stationary</u> velocity field, the <u>trajectories and the streamlines coincide</u> – PROOF:
  2. If $\mathbf{v}(\mathbf{x},t)=\mathbf{v}(\mathbf{x})$ the envelopes (i.e., the streamline) of the field do not vary throughout time.

     ✚

     A particle's trajectory is always tangent to the velocity field it encounters at every time instant.

     ⇓

     If a trajectory starts at a certain point in a streamline and the streamline does not vary with time and neither does the velocity field, the trajectory and streamline MUST coincide.

Trajectories and Streamlines

- The inverse is not necessarily true: if the <u>trajectories and the streamlines coincide</u>, the velocity field is **not** necessarily <u>stationary</u> – COUNTER-EXAMPLE:
  - Given the (non-stationary) velocity field: $\left[\mathbf{v}(\mathrm{t})\right] = \begin{bmatrix} at \\ 0 \\ 0 \end{bmatrix}$
  - The eq. trajectory are:

$$\left[\frac{d\mathbf{x}(\mathrm{t})}{dt}\right] = \begin{bmatrix} at \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[d\mathbf{x}(\mathrm{t})\right] = \begin{bmatrix} at \\ 0 \\ 0 \end{bmatrix} dt \Rightarrow \boxed{\mathbf{x}(\mathrm{t}) = \begin{bmatrix} a/2\, t^2 + C_1 \\ 0 \\ 0 \end{bmatrix}}$$

  - The eq. streamlines are:

$$\left[\frac{d\mathbf{x}(\lambda)}{d\lambda}\right] = \begin{bmatrix} at \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[d\mathbf{x}(\lambda)\right] = \begin{bmatrix} at \\ 0 \\ 0 \end{bmatrix} d\lambda \Rightarrow \boxed{\mathbf{x}(\mathrm{t}) = \begin{bmatrix} at\lambda + C_1' \\ 0 \\ 0 \end{bmatrix}}$$

Example

Consider the following velocity field:

$$\mathrm{v}_i = \frac{x_i}{1+t} \qquad i \in \{1,2,3\}$$

Obtain the equation of the trajectories and the streamlines associated to this vector field.

Do they coincide? Why?

Example - Solution

Eq. trajectories:

$$\begin{cases} \dfrac{d\mathbf{x}(t)}{dt} = \mathbf{v}\big(\mathbf{x}(t),t\big) \\ \dfrac{dx_i(t)}{dt} = \mathrm{v}_i\big(\mathbf{x}(t),t\big) \quad i \in \{1,2,3\} \end{cases}$$

Introducing the velocity field and rearranging:

$$\frac{dx_i}{dt} = \frac{x_i}{1+t} \quad i \in \{1,2,3\} \quad \Rightarrow \quad \frac{dx_i}{x_i} = \frac{dt}{1+t} \quad i \in \{1,2,3\}$$

Integrating both sides of the expression:

$$\int \frac{1}{x_i} dx_i = \int \frac{1}{1+t} dt \quad \Rightarrow \quad \ln x_i = \ln(1+t) + \ln C_i = \ln C_i (1+t) \quad i \in \{1,2,3\}$$

The solution:

$$\boxed{x_i = C_i (1+t) \quad i \in \{1,2,3\}}$$

Example - Solution

Eq. streamlines:

$$\begin{cases} \dfrac{d\mathbf{x}(\lambda)}{d\lambda} = \mathbf{v}\left(\mathbf{x}(\lambda), t^*\right) \\ \dfrac{dx_i(t)}{d\lambda} = \mathrm{v}_i\left(\mathbf{x}(\lambda), t^*\right) \quad i \in \{1,2,3\} \end{cases}$$

Introducing the velocity field and rearranging:

$$\frac{dx_i}{d\lambda} = \frac{x_i}{1+t} \quad i \in \{1,2,3\} \quad \Rightarrow \quad \frac{dx_i}{x_i} = \frac{d\lambda}{1+t} \quad i \in \{1,2,3\}$$

Integrating both sides of the expression:

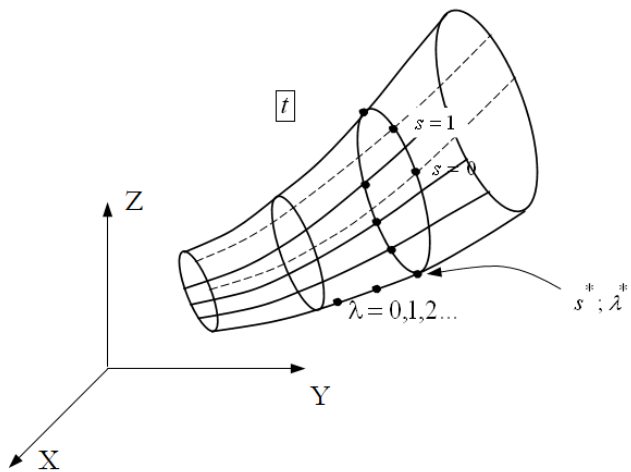
$$\int \frac{1}{x_i} dx_i = \int \frac{1}{1+t} d\lambda \quad \Rightarrow \quad \ln x_i = \frac{\lambda}{1+t} + K_i \quad \Rightarrow \quad \begin{cases} x_i = e^{\left(\frac{\lambda}{1+t} + K_i\right)} = \underbrace{e^{K_i}}_{= C_i} e^{\left(\frac{\lambda}{1+t}\right)} \\ i \in \{1,2,3\} \end{cases}$$

The solution:

$$\boxed{x_i = C_i e^{\left(\frac{\lambda}{1+t}\right)} \quad i \in \{1,2,3\}}$$

Streamtube

- A **streamtube** is a surface composed of streamlines which pass through the points of a closed contour fixed in space.



  - In stationary cases, the tube will remain fixed in space throughout time. In non-stationary cases, it will vary (although the closed contour line is fixed).

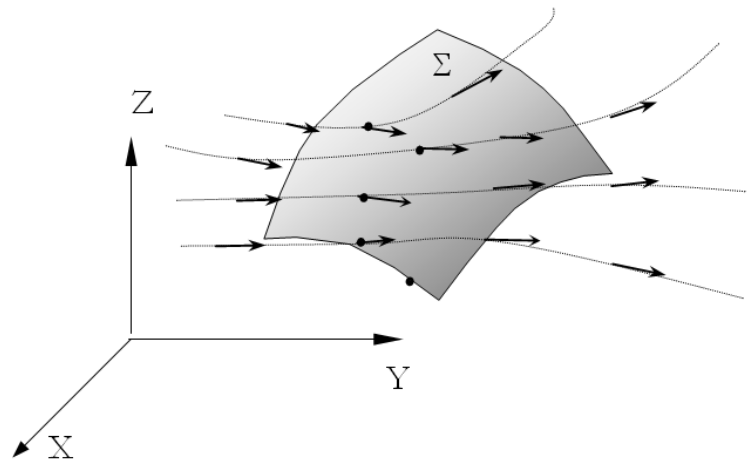
1.9 Control and Material Surfaces

Ch.1. Description of Motion

Control Surface

- A **control surface** is a fixed surface in space which does not vary in time.

$$\Sigma := \{ \mathrm{x} \mid f(x, y, z) = 0 \}$$

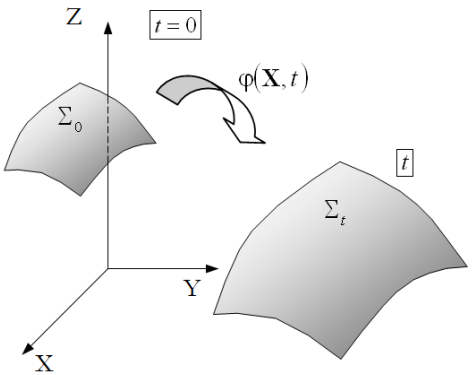


- Mass (particles) can flow across a control surface.

Material Surface

- A **material surface** is a mobile surface in the space constituted always by the same particles.
  - In the **reference configuration,** the surface Σ_0 will be defined in terms of the material coordinates:

$$\Sigma_0 := \{ \mathbf{X} \mid F(X,Y,Z) = 0 \}$$



    - The set of particles (material points) belonging the surface are the same at all times

  - In **spatial description** $F(X,Y,Z) = F(X(\mathbf{x},t),Y(\mathbf{x},t),Z(\mathbf{x},t)) = f(\mathbf{x},t) = f(x,y,z,t)$

$$\Sigma_t := \{ \mathbf{x} \mid f(x,y,z,t) = 0 \}$$

    - The set of spatial points belonging to the the surface depends on time
    - The material surface **moves in space**

Material Surface

- <u>Necessary and sufficient condition</u> for a mobile surface in space, implicitly defined by the function $f(x,y,z,t)$, to be a material surface is that the **material derivative** of the function **is zero:**
  - *Necessary*: if it is a material surface, its material description does not depend on time:

$$f(\mathbf{x},t) \rightarrow f(\mathbf{x}(\mathbf{X},t),t) = F(\mathbf{X},\cancel{t}) \Rightarrow \quad 0 = \frac{d}{dt} f(\mathbf{x},t) = \frac{\partial F(\mathbf{X})}{\partial t} = 0$$

  - *Sufficient*: if the material derivative of $f(\mathbf{x},t)$ is null:

$$f(\mathbf{x},t) \rightarrow f(\mathbf{x}(\mathbf{X},t),t) = F(\mathbf{X},\cancel{t}) \Rightarrow 0 = \frac{d}{dt} f(\mathbf{x},t) = \frac{\partial F(\mathbf{X},t)}{\partial t} \Rightarrow F(\mathbf{X},t) \equiv F(\mathbf{X})$$

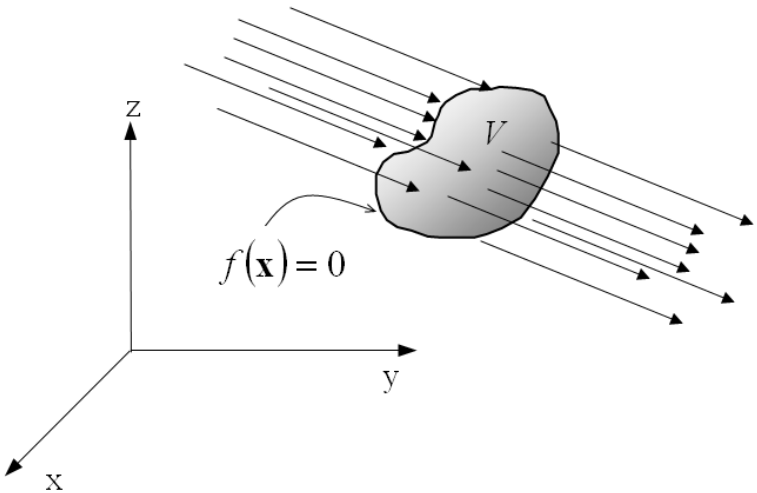
The surface $\Sigma_t := \{ \mathbf{x} \mid f(\mathbf{x},t) = 0 \} = \{ \mathbf{X} \mid F(\mathbf{X}) = 0 \}$

contains always the same set the of particles (it is a material surface)

Control Volume

- A **control volume** is a group of fixed points in space situated in the interior of a closed control surface, which does not vary in time.

$$V := \{ \mathbf{x} \quad | \quad f(\mathbf{x}) \leq 0 \}$$



REMARK

The function $f(\mathbf{x})$ is defined so that $f(\mathbf{x})<0$ corresponds to the points inside V.

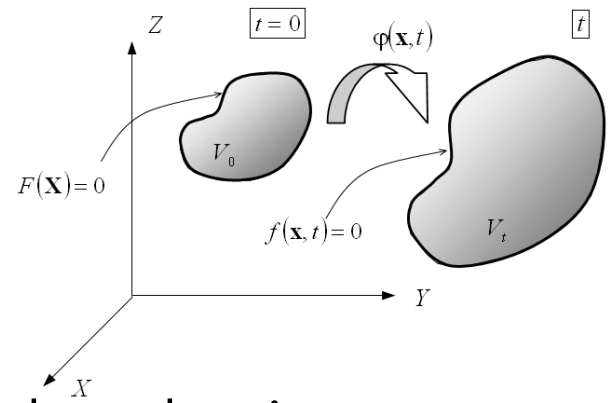
- Particles can enter and exit a control volume.

Material Volume

- A **material volume** is a (mobile) volume enclosed inside a material boundary or surface.
  - In the **reference configuration,** the volume V_0 will be defined in terms of the material coordinates:

$$V_0 := \{ \mathbf{X} \quad | \quad F(\mathbf{X}) \leq 0 \}$$

    - The particles $\mathbf{X}$ in the volume are the same at all times



  - In **spatial description,** the volume V_t will depend on time.

$$V_t := \{ \mathbf{x} \quad | \quad f(\mathbf{x},t) \leq 0 \}$$

    - The set of spatial points belonging to the the volume depends on time
    - The material volume **moves in space along time**

Material Volume

- A **material volume** is always constituted by the same particles. This is proved by *reductio ad absurdum*:
  - If a particle is added into the volume, it would have to cross its material boundary.
  - Material boundaries are constituted always by the same particles, so, no particles can cross.
  - Thus, a material volume is always constituted by the same particles

(a material volume is a pack of particles).

Chapter 1
Description of Motion

1.1 Definition of the Continuous Medium

A *continuous medium* is understood as an infinite set of particles (which form part of, for example, solids or fluids) that will be studied macroscopically, that is, without considering the possible discontinuities existing at microscopic level (atomic or molecular level). Accordingly, one admits that there are no discontinuities between the particles and that the mathematical description of this medium and its properties can be described by continuous functions.

1.2 Equations of Motion

The most basic description of the motion of a continuous medium can be achieved by means of mathematical functions that describe the position of each particle along time. In general, these functions and their derivatives are required to be continuous.

Definition 1.1. Consider the following definitions:

- *Spatial point*: Fixed point in space.
- *Material point*: A particle. It may occupy different spatial points during its motion along time.
- *Configuration*: Locus of the positions occupied in space by the particles of the continuous medium at a given time t.

The continuous medium is assumed to be composed of an infinite number of particles (*material points*) that occupy different positions in the physical space during its motion along time (see Figure 1.1). The *configuration* of the contin-

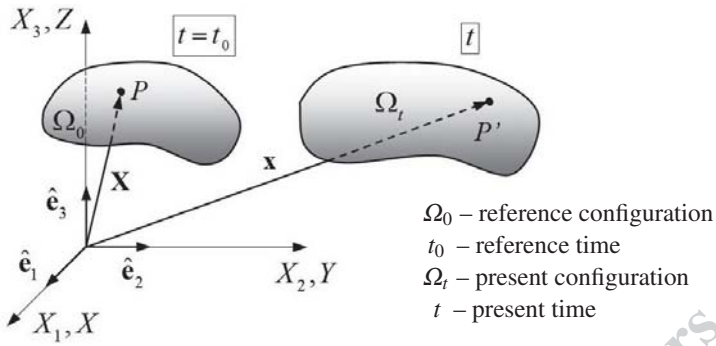


Figure 1.1: Configurations of the continuous medium.

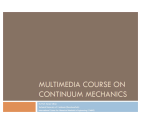
uous medium at time t, denoted by Ω_t, is defined as the locus of the positions occupied in space by the material points (particles) of the continuous medium at the given time.

A certain time $t = t_0$ of the time interval of interest is referred to as the *reference time* and the configuration at this time, denoted by Ω_0, is referred to as *initial*, *material* or *reference configuration*[1].

Consider now the Cartesian coordinate system (X, Y, Z) in Figure 1.1 and the corresponding orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. In the reference configuration Ω_0, the position vector $\mathbf{X}$ of a particle occupying a point P in space (at the reference time) is given by[2,3]

$$\mathbf{X} = X_1\hat{\mathbf{e}}_1 + X_2\hat{\mathbf{e}}_2 + X_3\hat{\mathbf{e}}_3 = X_i\hat{\mathbf{e}}_i, \tag{1.1}$$

where the components (X_1, X_2, X_3) are referred to as *material coordinates* (of the particle) and can be collected in a vector of components denoted as[4]



$$\mathbf{X} \overset{not}{\equiv} [\mathbf{X}] = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \overset{def}{=} \text{material coordinates.} \tag{1.2}$$

[1] In general, the time $t_0 = 0$ will be taken as the reference time.

[2] Notations (X, Y, Z) and (X_1, X_2, X_3) will be used indistinctly to designate the Cartesian coordinate system.

[3] *Einstein* or *repeated index notation* will be used in the remainder of this text. Every repetition of an index in the *same* monomial of an algebraic expression represents the sum over that index. For example,

$$\sum_{i=1}^{i=3} X_i\hat{\mathbf{e}}_i \overset{not}{=} X_i\hat{\mathbf{e}}_i \quad , \quad \sum_{k=1}^{k=3} a_{ik}b_{kj} \overset{not}{=} a_{ik}b_{kj} \quad \text{and} \quad \sum_{i=1}^{i=3}\sum_{j=1}^{j=3} a_{ij}b_{ij} \overset{not}{=} a_{ij}b_{ij}.$$

[4] Here, the vector (physical entity) $\mathbf{X}$ is distinguished from its vector of components $[\mathbf{X}]$. Henceforth, the symbol $\overset{not}{\equiv}$ (equivalent notation) will be used to indicate that the tensor and component notations at either side of the symbol are equivalent when the system of coordinates used remains unchanged.

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In the present configuration Ω_t [5], a particle originally located at a material point P (see Figure 1.1) occupies a spatial point P' and its position vector $\mathbf{x}$ is given by

$$\mathbf{x} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 = x_i\hat{\mathbf{e}}_i, \tag{1.3}$$

where (x_1, x_2, x_3) are referred to as *spatial coordinates* of the particle at time t,

$$\mathbf{x} \overset{not}{=} [\mathbf{x}] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{def}{=} \text{spatial coordinates.} \tag{1.4}$$

The motion of the particles of the continuous medium can now be described by the evolution of their spatial coordinates (or their position vector) along time. Mathematically, this requires the definition of a function that provides for each particle (identified by its *label*) its spatial coordinates x_i (or its spatial position vector $\mathbf{x}$) at successive instants of time. The material coordinates X_i of the particle can be chosen as the label that univocally characterizes it and, thus, the *equation of motion*

$$\begin{cases} \mathbf{x} = \varphi(\text{particle}, t) = \varphi(\mathbf{X}, t) \overset{not}{=} \mathbf{x}(\mathbf{X}, t) \\ x_i = \varphi_i(X_1, X_2, X_3, t) \quad i \in \{1, 2, 3\} \end{cases} \tag{1.5}$$

is obtained, which provides the spatial coordinates in terms of the material ones. The spatial coordinates x_i of the particle can also be chosen as label, defining the *inverse equation of motion*[6] as

$$\begin{cases} \mathbf{X} = \varphi^{-1}(\mathbf{x}, t) \overset{not}{=} \mathbf{X}(\mathbf{x}, t)\,, \\ X_i = \varphi_i^{-1}(x_1, x_2, x_3, t) \quad i \in \{1, 2, 3\}\,, \end{cases} \tag{1.6}$$

which provides the material coordinates in terms of the spatial ones.

Remark 1.1. There are different alternatives when choosing the label that characterizes a particle, even though the option of using its material coordinates is the most common one. When the equation of motion is written in terms of the material coordinates as label (as in (1.5)), one refers to it as the *equation of motion in canonical form*.

[5] Whenever possible, uppercase letters will be used to denote variables relating to the reference configuration Ω_0 and lowercase letters to denote the variables referring to the current configuration Ω_t.

[6] With certain abuse of notation, the function will be frequently confused with its image. Hence, the equation of motion will be often written as $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ and its inverse equation as $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$.

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There exist certain mathematical restrictions to guarantee the existence of φ and φ^{-1}, as well as their correct physical meaning. These restrictions are:

- $\varphi(\mathbf{X},0) = \mathbf{X}$ since, by definition, $\mathbf{X}$ is the position vector at the reference time $t = 0$ (consistency condition).
- $\varphi \in C^1$ (function φ is continuous with continuous derivatives at each point and at each instant of time).
- φ is biunivocal (to guarantee that two particles do not occupy simultaneously the same point in space and that a particle does not occupy simultaneously more than one point in space).
- The Jacobian of the transformation $J = \det\left[\dfrac{\partial\varphi(\mathbf{X},t)}{\partial\mathbf{X}}\right] \overset{not}{=} \left|\dfrac{\partial\varphi(\mathbf{X},t)}{\partial\mathbf{X}}\right| > 0.$

The physical interpretation of this condition (which will be studied later) is that every differential volume must always be positive or, using the *principle of mass conservation* (which will be seen later), the density of the particles must always be positive.

Remark 1.2. The equation of motion at the reference time $t = 0$ results in $\mathbf{x}(\mathbf{X},t)|_{t=0} = \mathbf{X}$. Accordingly, $x = X$, $y = Y$, $z = Z$ is the equation of motion at the reference time and the Jacobian at this instant of time is[7]

$$J(\mathbf{X},0) = \left|\frac{\partial(xyz)}{\partial(XYZ)}\right| = \det\left[\frac{\partial x_i}{\partial X_j}\right] = \det[\delta_{ij}] = \det\mathbf{1} = 1.$$

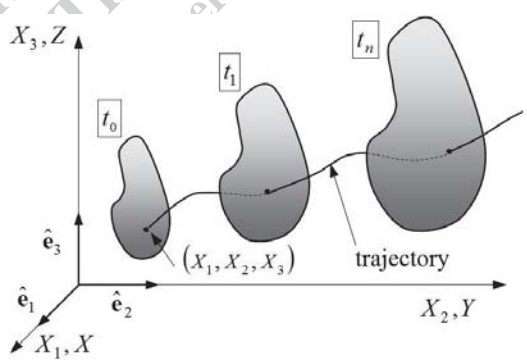

Figure 1.2: Trajectory or pathline of a particle.

[7] The two-index operator Delta Kronecker $\overset{not}{=} \delta_{ij}$ is defined as $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ij} = 1$ when $i = j$. Then, the unit tensor $\mathbf{1}$ is defined as $[\mathbf{1}]_{ij} = \delta_{ij}$.

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Remark 1.3. The expression $\mathbf{x} = \varphi(\mathbf{X}, t)$, particularized for a fixed value of the material coordinates $\mathbf{X}$, provides the equation of the *trajectory* or *pathline* of a particle (see Figure 1.2).

Example 1.1 – *The spatial description of the motion of a continuous medium is given by*

$$\mathbf{x}(\mathbf{X}, t) \overset{not}{\equiv} \begin{bmatrix} x_1 = X_1 \mathrm{e}^{2t} \\ x_2 = X_2 \mathrm{e}^{-2t} \\ x_3 = 5X_1 t + X_3 \mathrm{e}^{2t} \end{bmatrix} = \begin{bmatrix} x = X\mathrm{e}^{2t} \\ y = Y\mathrm{e}^{-2t} \\ z = 5Xt + Z\mathrm{e}^{2t} \end{bmatrix}$$

Obtain the inverse equation of motion.

Solution

The determinant of the Jacobian is computed as

$$J = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} \mathrm{e}^{2t} & 0 & 0 \\ 0 & \mathrm{e}^{-2t} & 0 \\ 5t & 0 & \mathrm{e}^{2t} \end{vmatrix} = \mathrm{e}^{2t} \neq 0.$$

The sufficient (but not necessary) condition for the function $\mathbf{x} = \varphi(\mathbf{X}, t)$ to be biunivocal (that is, for its inverse to exist) is that the determinant of the Jacobian of the function is not null. In addition, since the Jacobian is positive, the motion has physical sense. Therefore, the inverse of the given spatial description exists and is determined by

$$\mathbf{X} = \varphi^{-1}(\mathbf{x}, t) \overset{not}{\equiv} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} x_1 \mathrm{e}^{-2t} \\ x_2 \mathrm{e}^{2t} \\ x_3 \mathrm{e}^{-2t} - 5t x_1 \mathrm{e}^{-4t} \end{bmatrix}.$$

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1.3 Descriptions of Motion

The mathematical description of the properties of the particles of the continuous medium can be addressed in two alternative ways: the *material* description (typically used in solid mechanics) and the *spatial* description (typically used in fluid mechanics). Both descriptions essentially differ in the type of argument (material coordinates or spatial coordinates) that appears in the mathematical functions that describe the properties of the continuous medium.

1.3.1 Material Description

In the material description[8], a given property (for example, the density ρ) is described by a certain function $\overline{\rho}(\bullet,t):\ R^3 \times R^+ \to R^+$, where the argument $(\bullet)$ in $\overline{\rho}(\bullet,t)$ represents the material coordinates,

$$\rho = \overline{\rho}(\mathbf{X},t) = \overline{\rho}(X_1,X_2,X_3,t). \tag{1.7}$$

Here, if the three arguments $\mathbf{X} \equiv (X_1,X_2,X_3)$ are fixed, a specific particle is being followed (see Figure 1.3) and, hence, the name of material description.

1.3.2 Spatial Description

In the spatial description[9], the focus is on a point in space. The property is described as a function $\rho(\bullet,t):\ R^3 \times R^+ \to R^+$ of the point in space and of time,

$$\rho = \rho(\mathbf{x},t) = \rho(x_1,x_2,x_3,t). \tag{1.8}$$

Then, when the argument $\mathbf{x}$ in $\rho = \rho(\mathbf{x},t)$ is assigned a certain value, the evolution of the density for the *different* particles that occupy the point in space along time is obtained (see Figure 1.3). Conversely, fixing the time argument in (1.8) results in an instantaneous distribution (like a *snapshot*) of the property in space. Obviously, the direct and inverse equations of motion allow shifting from one

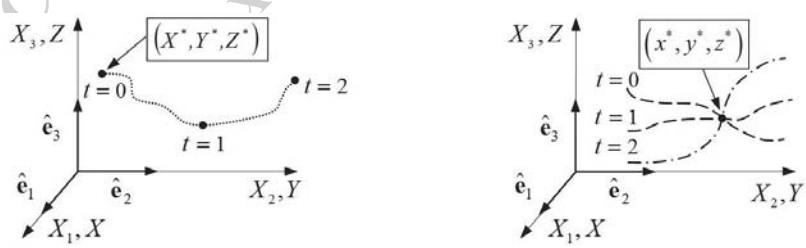

Figure 1.3: Material description (left) and spatial description (right) of a property.

[8] Literature on this topic also refers to the material description as *Lagrangian description*.
[9] The spatial description is also referred to as *Eulerian description*.

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description to the other as follows.

$$\begin{cases} \rho(\mathbf{x},t) = \rho(\mathbf{x}(\mathbf{X},t),t) = \overline{\rho}(\mathbf{X},t) \\ \overline{\rho}(\mathbf{X},t) = \overline{\rho}(\mathbf{X}(\mathbf{x},t),t) = \rho(\mathbf{x},t) \end{cases} \tag{1.9}$$

Example 1.2 – *The equation of motion of a continuous medium is*

$$\mathbf{x} = \mathbf{x}(\mathbf{X},t) \overset{not}{\equiv} \begin{bmatrix} x = X - Yt \\ y = Xt + Y \\ z = -Xt + Z \end{bmatrix}.$$

Obtain the spatial description of the property whose material description is

$$\overline{\rho}(X,Y,Z,t) = \frac{X+Y+Z}{1+t^2}.$$

Solution

The equation of motion is given in the canonical form since in the reference configuration Ω_0 its expression results in

$$\mathbf{x} = \mathbf{X}(\mathbf{X},0) \overset{not}{\equiv} \begin{bmatrix} x = X \\ y = Y \\ z = Z \end{bmatrix}.$$

The determinant of the Jacobian is

$$J = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \dfrac{\partial x}{\partial X} & \dfrac{\partial x}{\partial Y} & \dfrac{\partial x}{\partial Z} \\ \dfrac{\partial y}{\partial X} & \dfrac{\partial y}{\partial Y} & \dfrac{\partial y}{\partial Z} \\ \dfrac{\partial z}{\partial X} & \dfrac{\partial z}{\partial Y} & \dfrac{\partial z}{\partial Z} \end{vmatrix} = \begin{vmatrix} 1 & -t & 0 \\ t & 1 & 0 \\ -t & 0 & 1 \end{vmatrix} = 1 + t^2 \neq 0$$

and the inverse equation of motion is given by

$$\mathbf{X}(\mathbf{x},t) \overset{not}{\equiv} \begin{bmatrix} X = \dfrac{x+yt}{1+t^2} \\ Y = \dfrac{y-xt}{1+t^2} \\ Z = \dfrac{z+zt^2+xt+yt^2}{1+t^2} \end{bmatrix}.$$

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Consider now the material description of the property,

$$\overline{\rho}\,(X,Y,Z,t) = \frac{X+Y+Z}{1+t^2},$$

its spatial description is obtained by introducing the inverse equation of motion into the expression above,

$$\overline{\rho}\,(X,Y,Z,t) \equiv \frac{x+yt+y+z+zt^2+yt^2}{(1+t^2)^2} = \rho\,(x,y,z,t)\,.$$

1.4 Time Derivatives: Local, Material and Convective

The consideration of different descriptions (material and spatial) of the properties of the continuous medium leads to diverse definitions of the time derivatives of these properties. Consider a certain property and its material and spatial descriptions,

$$\Gamma\,(\mathbf{X},t) = \gamma(\mathbf{x},t)\,, \tag{1.10}$$

in which the change from the spatial to the material description and vice versa is performed by means of the equation of motion (1.5) and its inverse equation (1.6).

Definition 1.2. The *local derivative* of a property is its variation along time at a fixed point in space. If the *spatial description* $\gamma(\mathbf{x},t)$ of the property is available, the local derivative is mathematically written as[10]

$$\text{local derivative} \overset{not}{=} \frac{\partial\gamma(\mathbf{x},t)}{\partial t}\,.$$

The *material derivative* of a property is its variation along time following a specific particle (material point) of the continuous medium. If the *material description* $\Gamma\,(\mathbf{X},t)$ of the property is available, the material derivative is mathematically written as

$$\text{material derivative} \overset{not}{=} \frac{d}{dt}\Gamma = \frac{\partial\Gamma\,(\mathbf{X},t)}{\partial t}\,.$$

[10] The expression $\partial\,(\bullet,t)/\partial t$ is understood in the classical sense of partial derivative with respect to the variable t.

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However, taking the spatial description of the property $\gamma(\mathbf{x},t)$ and considering the equation of motion is implicit in this expression yields

$$\gamma(\mathbf{x},t) = \gamma(\mathbf{x}(\mathbf{X},t),t) = \Gamma(\mathbf{X},t) \ . \tag{1.11}$$

Then, the material derivative (following a particle) is obtained from the *spatial description* of the property as

$$\text{material derivative} \overset{not}{=} \frac{d}{dt}\gamma(\mathbf{x}(\mathbf{X},t),t) = \frac{\partial \Gamma(\mathbf{X},t)}{\partial t} \ . \tag{1.12}$$

Expanding (1.12) results in[11]

$$\begin{aligned} \frac{d\gamma(\mathbf{x}(\mathbf{X},t),t)}{dt} &= \frac{\partial \gamma(\mathbf{x},t)}{\partial t} + \frac{\partial \gamma}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial \gamma(\mathbf{x},t)}{\partial t} + \frac{\partial \gamma}{\partial \mathbf{x}} \cdot \underbrace{\frac{\partial \mathbf{x}}{\partial t}}_{\mathbf{v}(\mathbf{x},t)} = \\ &= \frac{\partial \gamma(\mathbf{x},t)}{\partial t} + \frac{\partial \gamma}{\partial \mathbf{x}} \cdot \mathbf{v}(\mathbf{x},t) \ , \end{aligned} \tag{1.13}$$

where the definition of velocity as the derivative of the equation of motion (1.5) with respect to time has been taken into account,

$$\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \mathbf{V}(\mathbf{X}(\mathbf{x},t),t) = \mathbf{v}(\mathbf{x},t) \ . \tag{1.14}$$

The deduction of the material derivative from the spatial description can be generalized for any property $\chi(\mathbf{x},t)$ (of scalar, vectorial or tensorial character) as[12]

$$\underbrace{\frac{d\chi(\mathbf{x},t)}{dt}}_{\substack{\text{material}\\ \text{derivative}}} = \underbrace{\frac{\partial \chi(\mathbf{x},t)}{\partial t}}_{\substack{\text{local}\\ \text{derivative}}} + \underbrace{\mathbf{v}(\mathbf{x},t)\cdot\nabla\chi(\mathbf{x},t)}_{\substack{\text{convective}\\ \text{derivative}}} \ . \tag{1.15}$$

Remark 1.4. The expression in (1.15) implicitly defines the *convective derivative* $\mathbf{v}\cdot\nabla(\bullet)$ as the difference between the material and spatial derivatives of the property. In continuum mechanics, the term *convection* is applied to phenomena that are related to mass (or particle) transport. Note that, if there is no convection ($\mathbf{v} = \mathbf{0}$), the convective derivative disappears and the local and material derivatives coincide.

[11] In literature, the notation $D(\bullet)/Dt$ is often used as an alternative to $d(\bullet)/dt$.

[12] The symbolic form of the *spatial Nabla* operator, $\nabla \equiv \partial \hat{\mathbf{e}}_i/\partial x_i$, is considered here.

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Example 1.3 – *Given the equation of motion*

$$\mathbf{x}(\mathbf{X},t) \overset{not}{=} \begin{bmatrix} x = X + Yt + Zt \\ y = Y + 2Zt \\ z = Z + 3Xt \end{bmatrix} ,$$

and the spatial description of a property, $\rho(\mathbf{x},t) = 3x + 2y + 3t$, *obtain the material derivative of this property.*

Solution

The material description of the property is obtained introducing the equation of motion into its spatial description,

$$\overline{\rho}(X,Y,Z,t) = 3(X + Yt + Zt) + 2(Y + 2Zt) + 3t = 3X + 3Yt + 7Zt + 2Y + 3t .$$

The material derivative is then calculated as the derivative of the material description with respect to time,

$$\frac{\partial \overline{\rho}}{\partial t} = 3Y + 7Z + 3 .$$

An alternative way of deducing the material derivative is by using the concept of material derivative of the spatial description of the property,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \quad \text{with}$$

$$\frac{\partial \rho}{\partial t} = 3 , \quad \mathbf{v} = \frac{\partial \mathbf{x}}{\partial t} = [Y + Z,\ 2Z,\ 3X]^T \quad \text{and} \quad \nabla \rho = [3,\ 2,\ 0]^T .$$

Replacing in the expression of the material derivative operator,

$$\frac{d\rho}{dt} = 3 + 3Y + 7Z$$

is obtained. Note that the expressions for the material derivative obtained from the material description, $\partial \overline{\rho} / \partial t$, and the spatial description, $d\rho/dt$, coincide.

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1.5 Velocity and Acceleration

Definition 1.3. The *velocity* is the time derivative of the equation of motion.

The material description of velocity is, consequently, given by

$$\begin{cases} \mathbf{V}(\mathbf{X},t) = \dfrac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} \\ V_i(\mathbf{X},t) = \dfrac{\partial x_i(\mathbf{X},t)}{\partial t} \quad i \in \{1,\ 2,\ 3\} \end{cases} \tag{1.16}$$

and, if the inverse equation of motion $\mathbf{X} = \varphi^{-1}(\mathbf{x},t)$ is known, the spatial description of the velocity can be obtained as

$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X}(\mathbf{x},t),t) \ . \tag{1.17}$$

Definition 1.4. The *acceleration* is the time derivative of the velocity field.

If the velocity is described in material form, the material description of the acceleration is given by

$$\begin{cases} \mathbf{A}(\mathbf{X},t) = \dfrac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t} \\ A_i(\mathbf{X},t) = \dfrac{\partial V_i(\mathbf{X},t)}{\partial t} \quad i \in \{1,\ 2,\ 3\} \end{cases} \tag{1.18}$$

and, through the inverse equation of motion $\mathbf{X} = \varphi^{-1}(\mathbf{x},t)$, the spatial description is obtained, $\mathbf{a}(\mathbf{x},t) = \mathbf{A}(\mathbf{X}(\mathbf{x},t),t)$. Alternatively, if the spatial description of the velocity is available, applying (1.15) to obtain the material derivative of $\mathbf{v}(\mathbf{x},t)$,

$$\mathbf{a}(\mathbf{x},t) = \frac{d\mathbf{v}(\mathbf{x},t)}{dt} = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t) \cdot \nabla \mathbf{v}(\mathbf{x},t) \ , \tag{1.19}$$

directly yields the spatial description of the acceleration.

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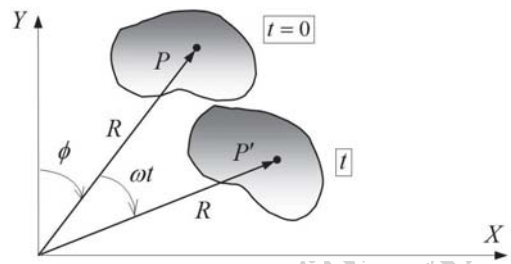
Example 1.4 – *Consider the solid in the figure below, which rotates at a constant angular velocity ω and has the expression*

$$\begin{cases} x = R\sin(\omega t + \phi) \\ y = R\cos(\omega t + \phi) \end{cases}$$

as its equation of motion. Find the velocity and acceleration of the motion described both in material and spatial forms.



Solution

The equation of motion can be rewritten as

$$\begin{cases} x = R\sin(\omega t + \phi) = R\sin(\omega t)\cos\phi + R\cos(\omega t)\sin\phi \\ y = R\cos(\omega t + \phi) = R\cos(\omega t)\cos\phi - R\sin(\omega t)\sin\phi \end{cases}$$

and, since for $t = 0$, $X = R\sin\phi$ and $Y = R\cos\phi$, the canonical form of the equation of motion and its inverse equation result in

$$\begin{cases} x = X\cos(\omega t) + Y\sin(\omega t) \\ y = -X\sin(\omega t) + Y\cos(\omega t) \end{cases} \quad \text{and} \quad \begin{cases} X = x\cos(\omega t) - y\sin(\omega t) \\ Y = x\sin(\omega t) + y\cos(\omega t) \end{cases} .$$

Velocity in material description:

$$\mathbf{V}(\mathbf{X},t) = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}) \overset{not}{\equiv} \begin{bmatrix} \dfrac{\partial x}{\partial t} = -X\omega\sin(\omega t) + Y\omega\cos(\omega t) \\ \dfrac{\partial y}{\partial t} = -X\omega\cos(\omega t) - Y\omega\sin(\omega t) \end{bmatrix}$$

Velocity in spatial description:
Replacing the canonical form of the equation of motion into the material description of the velocity results in

$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X}(\mathbf{x},t),t) \overset{not}{\equiv} \begin{bmatrix} \omega y \\ -\omega x \end{bmatrix} .$$

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Acceleration in material description:

$$\mathbf{A}(\mathbf{X},t) = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t}$$

$$\mathbf{A}(\mathbf{X},t) \overset{not}{\equiv} \begin{bmatrix} \dfrac{\partial \mathrm{v}_x}{\partial t} = -X\omega^2\cos(\omega t) - Y\omega^2\sin(\omega t) \\ \dfrac{\partial \mathrm{v}_y}{\partial t} = X\omega^2\sin(\omega t) - Y\omega^2\cos(\omega t) \end{bmatrix} =$$

$$= -\omega^2 \begin{bmatrix} X\cos(\omega t) + Y\sin(\omega t) \\ -X\sin(\omega t) + Y\cos(\omega t) \end{bmatrix}$$

Acceleration in spatial description:
Replacing the canonical form of the equation of motion into the material description of the acceleration results in

$$\mathbf{a}(\mathbf{x},t) = \mathbf{A}(\mathbf{X}(\mathbf{x},t),t) \overset{not}{\equiv} \begin{bmatrix} -\omega^2 x \\ -\omega^2 y \end{bmatrix}.$$

This same expression can be obtained if the expression for the velocity $\mathbf{v}(\mathbf{x},t)$ and the definition of material derivative in (1.15) are taken into account,

$$\mathbf{a}(\mathbf{x},t) = \frac{d\mathbf{v}(\mathbf{x},t)}{dt} = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla\mathbf{v}(\mathbf{x},t) =$$

$$\overset{not}{\equiv} \frac{\partial}{\partial t}\begin{bmatrix} \omega y \\ -\omega x \end{bmatrix} + \begin{bmatrix} \omega y\,, & -\omega x \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \omega y\,, & -\omega x \end{bmatrix},$$

$$\mathbf{a}(\mathbf{x},t) \overset{not}{\equiv} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \omega y\,, & -\omega x \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x}(\omega y) & \dfrac{\partial}{\partial x}(-\omega x) \\ \dfrac{\partial}{\partial y}(\omega y) & \dfrac{\partial}{\partial y}(-\omega x) \end{bmatrix} = \begin{bmatrix} -\omega^2 x \\ -\omega^2 y \end{bmatrix}.$$

Note that the result obtained using both procedures is identical.

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1.6 Stationarity

Definition 1.5. A property is *stationary* when its spatial description does not depend on time.

According to the above definition, and considering the concept of local derivative, any stationary property has a null local derivative. For example, if the velocity for a certain motion is stationary, it can be described in spatial form as

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{x}) \iff \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} = 0 . \tag{1.20}$$

Remark 1.5. The non-dependence on time of the spatial description (stationarity) assumes that, *for a same point in space*, the property being considered does not vary along time. This does not imply that, *for a same particle*, such property does not vary along time (the material description may depend on time). For example, if the velocity $\mathbf{v}(\mathbf{x},t)$ is stationary,

$$\mathbf{v}(\mathbf{x},t) \equiv \mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x}(\mathbf{X},t)) = \mathbf{V}(\mathbf{X},t) ,$$

and, thus, the material description of the velocity depends on time. In the case of stationary density (see Figure 1.4), for two particles labeled $\mathbf{X}_1$ and $\mathbf{X}_2$ that have varying densities along time, when occupying a same spatial point $\mathbf{x}$ (at two different times t_1 and t_2) their density value will coincide,

$$\overline{\rho}(\mathbf{X}_1,t_1) = \overline{\rho}(\mathbf{X}_2,t_2) = \rho(\mathbf{x}) .$$

That is, for an observer placed outside the medium, the density of the fixed point in space $\mathbf{x}$ will always be the same.

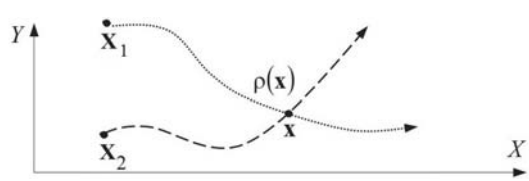

Figure 1.4: Motion of two particles with stationary density.

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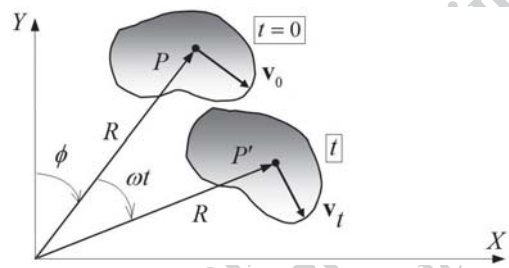
Example 1.5 – *Justify if the motion described in Example 1.4 is stationary or not.*

Solution

The velocity field in Example 1.4 is $\mathbf{v}(\mathbf{x}) \overset{not}{\equiv} [\omega y\ ,\ -\omega x]^T$. Therefore, it is a case in which the spatial description of the velocity is not dependent on time and, thus, the velocity is stationary. Obviously, this implies that the velocity of the particles (whose motion is a *uniform rotation* with respect to the origin, with angular velocity ω) does not depend on time (see figure below). The direction of the velocity vector for a same particle is tangent to its circular trajectory and changes along time.



The acceleration (material derivative of the velocity),

$$\mathbf{a}(\mathbf{x}) = \frac{d\mathbf{v}(\mathbf{x})}{dt} = \frac{\partial \mathbf{v}(\mathbf{x})}{\partial t} + \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x})\ ,$$

appears due to the change in direction of the velocity vector of the particles and is known as the *centripetal acceleration*.

1.7 Trajectory

Definition 1.6. A *trajectory* (or *pathline*) is the locus of the positions occupied in space by a given particle along time.

The parametric equation of a trajectory as a function of time is obtained by particularizing the equation of motion for a given particle (identified by its material coordinates $\mathbf{X}^*$, see Figure 1.5),

$$\mathbf{x}(t) = \varphi(\mathbf{X}, t)\Big|_{\mathbf{X}=\mathbf{X}^*}\ . \tag{1.21}$$

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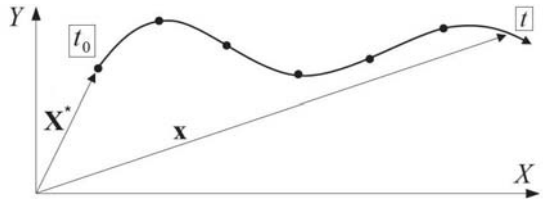

Figure 1.5: Trajectory or pathline of a particle.

Given the equation of motion $\mathbf{x} = \varphi(\mathbf{X}, t)$, each point in space is occupied by a trajectory characterized by the value of the label (material coordinates) $\mathbf{X}$. Then, the equation of motion defines a family of curves whose elements are the trajectories of the various particles.

1.7.1 Differential Equation of the Trajectories

Given the velocity field in spatial description $\mathbf{v}(\mathbf{x}, t)$, the family of trajectories can be obtained by formulating the system of differential equations that imposes that, for each point in space $\mathbf{x}$, the velocity vector is the time derivative of the parametric equation of the trajectory defined in (1.21), i.e.,

$$\text{Find } \mathbf{x}(t) := \begin{cases} \dfrac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t), t) \,, \\ \dfrac{dx_i(t)}{dt} = \mathrm{v}_i(x(t), t) \quad i \in \{1, 2, 3\} \,. \end{cases} \tag{1.22}$$

The solution to this first-order system of differential equations depends on three integration constants (C_1, C_2, C_3),

$$\begin{cases} \mathbf{x} = \phi(C_1, C_2, C_3, t) \,, \\ x = \phi_i(C_1, C_2, C_3, t) \quad i \in \{1, 2, 3\} \,. \end{cases} \tag{1.23}$$

These expressions constitute a family of curves in space parametrized by the constants (C_1, C_2, C_3). Assigning a particular value to these constants yields a member of the family, which is the trajectory of a particle characterized by the label (C_1, C_2, C_3).

To obtain the equation in canonical form, the consistency condition is imposed in the reference configuration,

$$\mathbf{x}(t)\Big|_{t=0} = \mathbf{X} \implies \mathbf{X} = \phi(C_1, C_2, C_3, 0) \implies C_i = \chi_i(\mathbf{X}) \;\; i \in \{1, 2, 3\}, \tag{1.24}$$

and, replacing into (1.23), the canonical form of the equation of the trajectory,

$$\mathbf{X} = \phi(C_1(\mathbf{X}), C_2(\mathbf{X}), C_3(\mathbf{X}), t) = \varphi(\mathbf{X}, t) \,, \tag{1.25}$$

is obtained.

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Example 1.6 – *Given the velocity field in Example 1.5,* $\mathbf{v}(\mathbf{x}) \overset{not}{=} [\omega y\,,\ -\omega x]^T$, *obtain the equation of the trajectory.*

Solution

Using expression (1.22), one can write

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x},t) \quad \Longrightarrow \quad \begin{cases} \dfrac{dx(t)}{dt} = \mathrm{v}_x(\mathbf{x},t) = \omega y\,, \\ \dfrac{dy(t)}{dt} = \mathrm{v}_y(\mathbf{x},t) = -\omega x\,. \end{cases}$$

This system of equations is a system with crossed variables. Differentiating the second equation and replacing the result obtained into the first equation yields

$$\frac{d^2y(t)}{dt^2} = -\omega\frac{dx(t)}{dt} = -\omega^2 y(t) \quad \Longrightarrow \quad y'' + \omega^2 y = 0\,.$$

The *characteristic equation* of this second-order differential equation is $r^2+\omega^2=0$ and its *characteristic solutions* are $r_j = \pm i\omega \quad j \in \{1,\,2\}$. Therefore, the y component of the equation of the trajectory is

$$y(t) = \text{Real Part}\left\{C_1 e^{iwt} + C_2 e^{-iwt}\right\} = C_1\cos(\omega t) + C_2\sin(\omega t)\ .$$

The solution for $x(t)$ is obtained from $dy/dt = -\omega x$, which results in $x = -dy/(\omega\, dt)$ and, therefore,

$$\begin{cases} x(C_1,C_2,t) = C_1\sin(\omega t) - C_2\cos(\omega t)\,, \\ y(C_1,C_2,t) = C_1\cos(\omega t) + C_2\sin(\omega t)\,. \end{cases}$$

This equation provides the expressions of the trajectories in a non-canonical form. The canonical form is obtained considering the initial condition,

$$\mathbf{x}(C_1,C_2,0) = \mathbf{X}\,,$$

that is,

$$\begin{cases} x(C_1,C_2,0) = -C_2 = X\,, \\ y(C_1,C_2,0) = C_1 = Y\,. \end{cases}$$

Finally, the equation of motion, or the equation of the trajectory, in canonical form

$$\begin{cases} x = Y\sin(\omega t) + X\cos(\omega t) \\ y = Y\cos(\omega t) - X\sin(\omega t) \end{cases}$$

is obtained.

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1.8 Streamline

Definition 1.7. The *streamlines* are a family of curves that, for every instant of time, are the velocity field envelopes[13].

According to its definition, the tangent at each point of a streamline has the same direction (though not necessarily the same magnitude) as the velocity vector at that same point in space.

Remark 1.6. In general, the velocity field (in spatial description) will be different for each instant of time ($\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$). Therefore, one must speak of a different family of streamlines for each instant of time (see Figure 1.6).

1.8.1 Differential Equation of the Streamlines

Consider a given time t^* and the spatial description of the velocity field at this time $\mathbf{v}(\mathbf{x}, t^*)$. Let $\mathbf{x}(\lambda)$ be the equation of a streamline parametrized in terms of a certain parameter λ. Then, the vector tangent to the streamline is defined, for

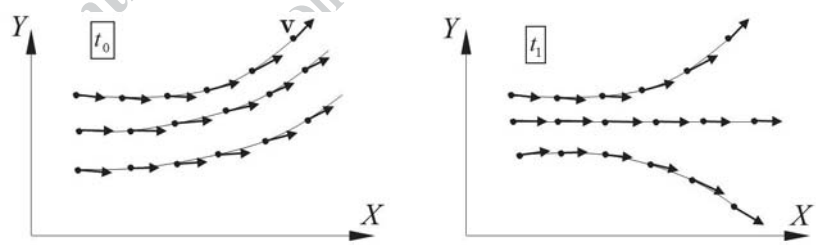

Figure 1.6: Streamlines at two different instants of time.

[13] The envelopes of a vector field are the family of curves whose tangent vector has, at each point, the same direction as the corresponding vector of the vector field.

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each value of λ [14], by $d\mathbf{x}(\lambda)/d\lambda$ and the vector field tangency condition can be written as follows.

$$\text{Find} \quad \mathbf{x}(\lambda) := \begin{cases} \dfrac{d\mathbf{x}(\lambda)}{d\lambda} = \mathbf{v}(\mathbf{x}(\lambda),\, t^*)\,, \\ \dfrac{dx_i(\lambda)}{d\lambda} = \mathrm{v}_i(\mathbf{x}(\lambda),\, t^*) \quad i \in \{1,\, 2,\, 3\}\,. \end{cases} \tag{1.26}$$

The expressions in (1.26) constitute a system of first-order differential equations whose solution for each time t^*, which will depend on three integration constants (C_1', C_2', C_3'), provides the parametric expression of the streamlines,

$$\begin{cases} \mathbf{x} = \boldsymbol{\phi}(C_1', C_2', C_3', \lambda, t^*)\,, \\ x_i = \phi_i(C_1', C_2', C_3', \lambda, t^*) \quad i \in \{1,\, 2,\, 3\}\,. \end{cases} \tag{1.27}$$

Each triplet of integration constants (C_1', C_2', C_3') identifies a streamline whose points, in turn, are obtained by assigning values to the parameter λ. For each time t^* a new family of streamlines is obtained.

Remark 1.7. In a stationary velocity field $(\mathbf{v}(\mathbf{x}, t) \equiv \mathbf{v}(\mathbf{x}))$ the *trajectories* and *streamlines coincide*. This can be proven from two different viewpoints:

- The fact that the time variable does not appear in (1.22) or (1.26) means that the differential equations defining the trajectories and those defining the streamlines only differ in the *denomination* of the integration parameter (t or λ, respectively). The solution to both systems must be, therefore, the same, except for the *name* of the parameter used in each type of curves.
- From a more physical point of view: a) If the velocity field is stationary, its envelopes (the streamlines) do not change along time; b) a given particle moves in space keeping the trajectory in the direction tangent to the velocity field it encounters along time; c) consequently, if a trajectory starts at a certain point in a streamline, it will stay on this streamline throughout time.

[14] It is assumed that the value of the parameter λ is chosen such that, at each point in space $\mathbf{x}$, not only does $d\mathbf{x}(\lambda)/d\lambda$ have the same direction as the vector $\mathbf{v}(\mathbf{x}, t)$, but it coincides therewith.

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1.9 Streamtubes

Definition 1.8. A *streamtube* is a surface formed by a bundle of streamlines that occupy the points of a closed line, fixed in space, and that *does not* constitute a streamline.

In non-stationary cases, even though the closed line does not vary in space, the streamtube and streamlines do change. On the contrary, in a stationary case, the streamtube remains fixed in space along time.

1.9.1 Equation of the Streamtube

Streamlines constitute a family of curves of the type

$$\mathbf{x} = \mathbf{f}(C_1, C_2, C_3, \lambda, t) \ . \tag{1.28}$$

The problem consists in determining, for each instant of time, which curves of the family of curves of the streamlines cross a closed line, which is fixed in the space Γ, whose mathematical expression parametrized in terms of a parameter s is

$$\Gamma := \mathbf{x} = \mathbf{g}(s) \ . \tag{1.29}$$

To this aim, one imposes, in terms of the parameters λ^* and s^*, that a same point belong to both curves,

$$\begin{cases} \mathbf{g}(s^*) = \mathbf{f}(C_1, C_2, C_3, \lambda^*, t) \ , \\ g_i(s^*) = f_i(C_1, C_2, C_3, \lambda^*, t) \quad i \in \{1,\ 2,\ 3\} \ . \end{cases} \tag{1.30}$$

A system of three equations is obtained from which, for example, s^*, λ^* and C_3 can be isolated,

$$\begin{aligned} s^* &= s^*(C_1, C_2, t) \ , \\ \lambda^* &= \lambda^*(C_1, C_2, t) \ , \\ C_3 &= C_3(C_1, C_2, t) \ . \end{aligned} \tag{1.31}$$

Introducing (1.31) into (1.30) yields

$$\mathbf{x} = \mathbf{f}(C_1,\ C_2,\ C_3(C_1, C_2, t),\ \lambda^*(C_1, C_2, t),\ t) = \mathbf{h}(C_1, C_2, t) \ , \tag{1.32}$$

which constitutes the parametrized expression (in terms of the parameters C_1 and C_2) of the streamtube for each time t (see Figure 1.7).

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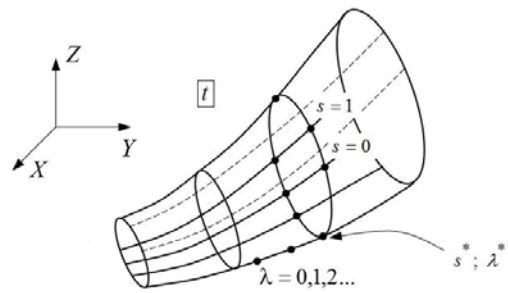

Figure 1.7: Streamtube at a given time t.

1.10 Streaklines

Definition 1.9. A *streakline*, relative to a fixed point in space $\mathbf{x}^*$ named *spill point* and at a time interval $[t_i, t_f]$ named *spill period*, is the locus of the positions occupied at time t by all the particles that have occupied $\mathbf{x}^*$ over the time $\tau \in [t_i, t] \bigcap [t_i, t_f]$.

The above definition corresponds to the physical concept of the color line (streak) that would be observed in the medium at time t if a tracer fluid were injected at spill point $\mathbf{x}^*$ throughout the time interval $[t_i, t_f]$ (see Figure 1.8).

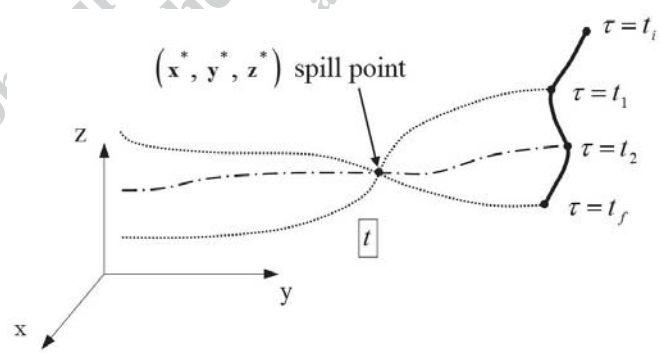

Figure 1.8: Streakline corresponding to the spill period $\tau \in [t_i, t_f]$.

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1.10.1 Equation of the Streakline

To determine the equation of a streakline one must identify all the particles that occupy point $\mathbf{x}^*$ in the corresponding times τ. Given the equation of motion (1.5) and its inverse equation (1.6), the label of the particle which at time τ occupies the spill point must be identified. Then,

$$\left.\begin{array}{l} \mathbf{x}^* = \mathbf{x}(\mathbf{X},\tau) \\ x_i^* = x_i(\mathbf{X},\tau) \quad i \in \{1,\,2,\,3\} \end{array}\right\} \implies \mathbf{X} = \mathbf{f}(\tau) \tag{1.33}$$

and replacing (1.33) into the equation of motion (1.5) results in

$$\mathbf{x} = \varphi(\mathbf{f}(\tau),t) = \mathbf{g}(\tau,t) \quad \tau \in [t_i,t] \bigcap [t_i,t_f] . \tag{1.34}$$

Expression (1.34) is, for each time t, the parametric expression (in terms of parameter τ) of a curvilinear segment in space which is the streakline at that time.

Example 1.7 – *Given the equation of motion*

$$\begin{cases} x = (X+Y)\,t^2 + X\cos t\,, \\ y = (X+Y)\cos t - X\,, \end{cases}$$

obtain the equation of the streakline associated with the spill point $\mathbf{x}^* = (0,1)$ *for the spill period* $[t_0,+\infty)$.

Solution

The material coordinates of a particle that has occupied the spill point at time τ are given by

$$\left.\begin{array}{l} 0 = (X+Y)\,\tau^2 + X\cos\tau \\ 1 = (X+Y)\cos\tau - X \end{array}\right\} \implies \begin{cases} X = \dfrac{-\tau^2}{\tau^2+\cos^2\tau}\,, \\ Y = \dfrac{\tau^2+\cos\tau}{\tau^2+\cos^2\tau}\,. \end{cases}$$

Therefore, the label of the particles that have occupied the spill point from the initial spill time t_0 until the present time t is defined by

$$\left.\begin{array}{l} X = \dfrac{-\tau^2}{\tau^2+\cos^2\tau} \\ Y = \dfrac{\tau^2+\cos\tau}{\tau^2+\cos^2\tau} \end{array}\right\} \quad \tau \in [t_0,t] \bigcap [t_0,\infty) = [t_0,t]\ .$$

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Then, replacing these into the equation of motion, the equation of the streakline is obtained,

$$\mathbf{x} = \mathbf{g}(\tau, t) \overset{not}{\equiv} \begin{bmatrix} x = \dfrac{\cos\tau}{\tau^2 + \cos^2\tau} t^2 + \dfrac{-\tau^2}{\tau^2 + \cos^2\tau} \cos t \\ y = \dfrac{\cos\tau}{\tau^2 + \cos^2\tau} \cos t - \dfrac{-\tau^2}{\tau^2 + \cos^2\tau} \end{bmatrix} \qquad \tau \in [t_0, t] \ .$$

Remark 1.8. In a *stationary* problem, the streaklines are segments of the trajectories (or of the streamlines). The rationale is based on the fact that, in the stationary case, the trajectory follows the envelope of the velocity field, which remains constant along time. If one considers a spill point $\mathbf{x}^*$, all the particles that occupy this point will follow portions (segments) of the same trajectory.

1.11 Material Surface

Definition 1.10. A *material surface* is a mobile surface in space always constituted by the same particles (material points).

In the reference configuration Ω_0, surface Σ_0 can be defined in terms of a function of the material coordinates $F(X,Y,Z)$ as

$$\Sigma_0 := \{X, Y, Z \mid F(X,Y,Z) = 0\} \ . \tag{1.35}$$

Remark 1.9. The function $F(X,Y,Z)$ does not depend on time, which guarantees that the particles, identified by their label, that satisfy equation $F(X,Y,Z) = 0$ are always the same in accordance with the definition of material surface.

The spatial description of the surface is obtained from the spatial description of $F(\mathbf{X}(\mathbf{x},t)) = f(x,y,z,t)$ as

$$\Sigma_t := \{x, y, z \mid f(x,y,z,t) = 0\} \ . \tag{1.36}$$

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Remark 1.10. The function $f(x,y,z,t)$ depends explicitly on time, which indicates that the *points in space* that are on the surface will vary along time. This time dependence of the spatial description of the surface confers the character of mobile surface in space to the surface (see Figure 1.9).

Remark 1.11. The necessary and sufficient condition for a mobile surface in space, defined implicitly by a function $f(x,y,z,t) = 0$, to be material (to be always constituted by the same particles) is that *the material derivative of* $f(x,y,z,t)$ *is null*,

$$\frac{df(\mathbf{x},t)}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = 0 \qquad \forall \mathbf{x} \in \Sigma_t \quad \forall t \ .$$

The condition is *necessary* because, if the surface is a material surface, its material description will not depend on time $(F \equiv F(\mathbf{X}))$ and, therefore, its spatial description will have a null material derivative. The condition of *sufficiency* is based on the fact that, if the material derivative of $f(\mathbf{x},t)$ is zero, the corresponding material description will not depend on time $(F \equiv F(\mathbf{X}))$ and, therefore, the set of particles (identified by their material coordinates) that satisfy the condition $F(\mathbf{X}) = 0$ is always the same.

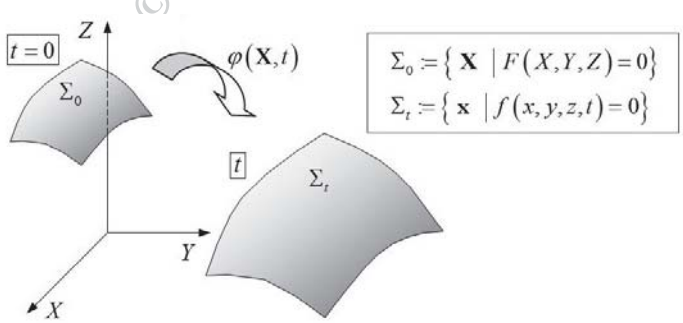

Figure 1.9: A material surface at two different instants of time.

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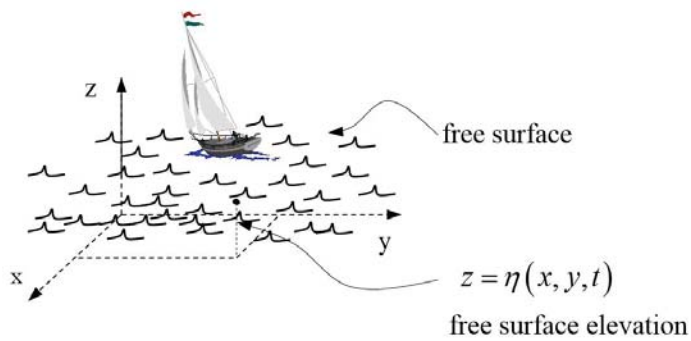
Example 1.8 – *In ocean waves theory, the condition that the free surface of the fluid in contact with the atmosphere is a material surface is imposed. This restriction implies that the free surface is always composed of the same particles, which is a reasonable hypothesis (especially in deep waters). Determine how this condition is stated in terms of the velocity field of the fluid.*

Solution

Assuming that $z = \eta\,(x,y,t)$ defines the elevation of the sea surface with respect to a reference level, the free surface of the water will be given by

$$f\,(x,y,z,t) \equiv z - \eta\,(x,y,t) = 0\,.$$



The condition $df/dt = 0$ can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f \quad \text{where} \quad \frac{\partial f}{\partial t} = -\frac{\partial \eta}{\partial t} \quad \text{and}$$

$$\mathbf{v}\cdot\nabla f \overset{not}{\equiv} [\mathrm{v}_x,\ \mathrm{v}_y,\ \mathrm{v}_z]\left[\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z}\right]^T = \mathrm{v}_x\frac{\partial f}{\partial x} + \mathrm{v}_y\frac{\partial f}{\partial y} + \mathrm{v}_z\frac{\partial f}{\partial z}\,.$$

Then,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f = -\frac{\partial \eta}{\partial t} - \mathrm{v}_x\frac{\partial \eta}{\partial x} - \mathrm{v}_y\frac{\partial \eta}{\partial y} + \mathrm{v}_z = 0$$

and, isolating v_z leads to

$$\mathrm{v}_z = \frac{\partial \eta}{\partial t} + \mathrm{v}_x\frac{\partial \eta}{\partial x} + \mathrm{v}_y\frac{\partial \eta}{\partial y}\,.$$

Therefore, the material surface condition results in a condition on the vertical component of the velocity field.

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1.12 Control Surface

Definition 1.11. A *control surface* is a fixed surface in space.

The mathematical description of a control surface is given by

$$\Sigma := \{\mathbf{x} \mid f(x,y,z) = 0\} \ . \tag{1.37}$$

Obviously, a control surface is occupied by the different particles of the continuous medium along time (see Figure 1.10).

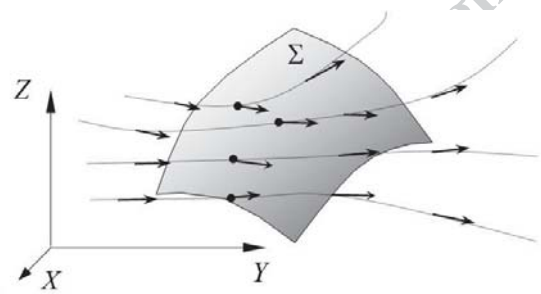

Figure 1.10: Movement of particles through a control surface along time.

1.13 Material Volume

Definition 1.12. A *material volume* is a volume enclosed by a closed material surface.

The mathematical description of a material volume (see Figure 1.11) is given, in the material description, by[15]

$$V_0 := \{\mathbf{X} \mid F(\mathbf{X}) \leq 0\} \tag{1.38}$$

and, in the spatial description, by

$$V_t := \{\mathbf{x} \mid f(\mathbf{x},t) \leq 0\}, \tag{1.39}$$

[15] It is assumed that function $F(\mathbf{X})$ is defined such that $F(\mathbf{X}) < 0$ corresponds to points in the interior of V_0.

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where $F(\mathbf{X}) = f(\mathbf{x}(\mathbf{X},t),t)$ is the function that describes the material surface that encloses the volume.

Remark 1.12. A material volume is always constituted by the same particles. This is proven by *reductio ad absurdum* as follows. If a certain particle could enter or exit the material volume, it would be incorporated into the material surface during its motion (at least, for an instant of time). This would be contrary to the fact that the surface, being a material surface, is always constituted by the same particles.

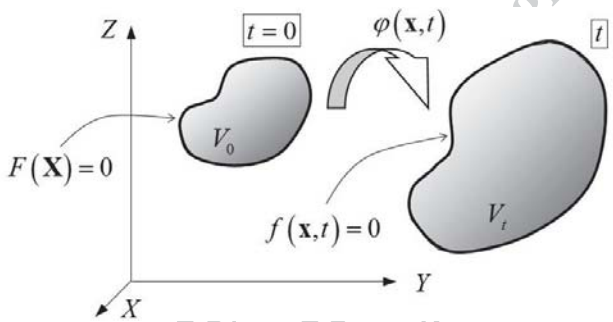

Figure 1.11: A material volume at two different instants of time.

1.14 Control Volume

Definition 1.13. A *control volume* is a group of points in space situated in the interior of a closed control surface.

It is a volume fixed in space that is occupied by the particles of the medium during its motion. The mathematical description of the control volume (see Figure 1.12) is[16]

$$V := \{\mathbf{x} \mid f(\mathbf{x}) \leq 0\} \ . \tag{1.40}$$

[16] It is assumed that function $f(\mathbf{x})$ is defined such that $f(\mathbf{x}) < 0$ corresponds to points in the interior of V.

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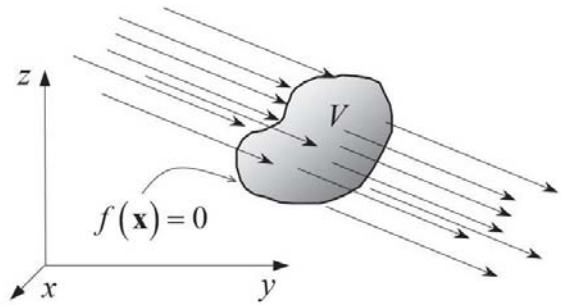

Figure 1.12: A control volume is occupied by different particles along time.

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PROBLEMS

Problem 1.1 – *Justify whether the following statements are true or false.*

a) If the velocity field is stationary, the acceleration field is also stationary.

b) If the velocity field is uniform, the acceleration field is always null.

Solution

a) A stationary velocity field implies that the spatial description of velocity does not depend on time,

$$\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} = 0 \quad \Longrightarrow \quad \mathbf{v}(\mathbf{x}) \ .$$

The acceleration is the material derivative of the velocity, therefore

$$\mathbf{a}(\mathbf{x},t) = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t) \cdot \nabla \mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x}) \ .$$

The resulting expression does not depend on time. Thus, *the statement is true*.

b) A uniform velocity field implies that the spatial description of velocity does not depend on the spatial coordinates,

$$\mathbf{v}(\mathbf{x},t) \quad \Longrightarrow \quad \mathbf{v}(t) \ .$$

The material derivative of the velocity results in

$$\mathbf{a}(\mathbf{x},t) = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t) \cdot \nabla \mathbf{v}(\mathbf{x},t) = \frac{\partial \mathbf{v}(t)}{\partial t} \ ,$$

where the expression used for the gradient of the velocity field is

$$[\nabla \mathbf{v}(t)]_{ij} = \frac{\partial \mathrm{v}_i(t)}{\partial \mathrm{x}_j} = 0 \ .$$

Therefore, *the statement is false* because $\partial \mathbf{v}(t)/\partial t$ is not necessarily zero.

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Problem 1.2 – *Calculate the acceleration at time $t = 2$ in point $(1,1,1)$ of the velocity field*

$$\mathbf{v} \overset{not}{\equiv} \left[x - z\,, \quad z\left(\mathrm{e}^{t} + \mathrm{e}^{-t}\right)\,, \quad 0\right]^{T}\,.$$

Solution

Since the velocity field is given in its spatial expression and the acceleration is requested for a point $\mathbf{x}^* = (1,1,1)^T$, the equation of motion is not needed. One can simply apply

$$\mathbf{a}\left(\mathbf{x},t\right) = \frac{d\mathbf{v}\left(\mathbf{x},t\right)}{dt} = \frac{\partial \mathbf{v}\left(\mathbf{x},t\right)}{\partial t} + \mathbf{v}\left(\mathbf{x},t\right)\cdot\nabla\mathbf{v}\left(\mathbf{x},t\right)\,,$$

where

$$\frac{\partial \mathbf{v}}{\partial t} \overset{not}{\equiv} \left[0\,, \quad z\left(\mathrm{e}^{t} - \mathrm{e}^{-t}\right)\,, \quad 0\right]^{T} \quad \text{and}$$

$$\nabla\mathbf{v} \overset{not}{\equiv} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \left[x - z\,, \quad z\left(\mathrm{e}^{t} + \mathrm{e}^{-t}\right), \quad 0\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & \left(\mathrm{e}^{t} + \mathrm{e}^{-t}\right) & 0 \end{bmatrix}, \text{ such that}$$

$$\mathbf{v}\cdot\nabla\mathbf{v} \overset{not}{\equiv} \left[x - z\,, \quad 0\,, \quad 0\right]^{T}\,.$$

Therefore, the spatial expression for the acceleration field is

$$\mathbf{a} \overset{not}{\equiv} \left[x - z\,, \quad z\left(\mathrm{e}^{t} - \mathrm{e}^{-t}\right)\,, \quad 0\right]^{T}$$

and, for the given point at the given instant of time, the acceleration is

$$\boxed{\mathbf{a}\left(\mathbf{x} = \mathbf{x}^*,\, t = 2\right) \overset{not}{\equiv} \left[0\,, \quad \mathrm{e}^{2} - \mathrm{e}^{-2}\,, \quad 0\right]^{T}}\,.$$

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Problem 1.3 – *The equation of a certain motion is*

$$x = X\,, \quad y = \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{t} + (Y-Z)\,\mathrm{e}^{-t}\right)\,, \quad z = \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{t} - (Y-Z)\,\mathrm{e}^{-t}\right)\,.$$

Calculate the accelerations that would be observed along time by:

a) An observer located in the fixed point $(1,1,1)$.

b) An observer traveling with the particle that at time $t=0$ *occupied position* $(1,1,1)$.

c) An observer located in point $(1,1,1)$ *that measures the accelerations as the difference between velocities at this point per unit of time.*

Solution

a) The spatial description of the acceleration in point $\mathbf{x}^* = (1,1,1)$ must be obtained,

$$\mathbf{a}\left(\mathbf{x}=\mathbf{x}^*,\, t\right) = \mathbf{A}\left(\mathbf{X}\left(\mathbf{x}^*,t\right),\, t\right) = \frac{\partial \mathbf{V}\left(\mathbf{X}\left(\mathbf{x}^*,t\right),t\right)}{\partial t}\,.$$

The material expression of the velocity field is

$$\mathbf{V}\left(\mathbf{X},t\right) = \frac{\partial \mathbf{x}\left(\mathbf{X},t\right)}{\partial t} \Longrightarrow \mathbf{V}\left(\mathbf{X},t\right) \overset{not}{\equiv} \begin{bmatrix} 0 \\ \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{t} - (Y-Z)\,\mathrm{e}^{-t}\right) \\ \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{t} + (Y-Z)\,\mathrm{e}^{-t}\right) \end{bmatrix}\,.$$

Then, the material description of the acceleration is

$$\mathbf{A}\left(\mathbf{X},t\right) = \frac{\partial \mathbf{V}\left(\mathbf{X},t\right)}{\partial t} \overset{not}{\equiv} \begin{bmatrix} 0 \\ \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{t} + (Y-Z)\,\mathrm{e}^{-t}\right) \\ \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{t} - (Y-Z)\,\mathrm{e}^{-t}\right) \end{bmatrix}\,.$$

Careful observation of the expression obtained reveals that

$$A_y = \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{t} + (Y-Z)\,\mathrm{e}^{-t}\right) = y \quad \text{and}$$

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$$A_z = \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{\,t} - (Y-Z)\,\mathrm{e}^{-t}\right) = z\,.$$

Therefore, the spatial description of the acceleration field is

$$\mathbf{a}\,(\mathbf{x},t) \overset{not}{\equiv} [0\,,\quad y\,,\quad z]^T$$

and, for $\mathbf{x} = \mathbf{x}^*$,

$$\boxed{\mathbf{a}\,(\mathbf{x}^*,t) \overset{not}{\equiv} [0\,,\ \ 1\,,\ \ 1]^T}\,.$$

NOTE: In case one does not realize that $A_y = y$ and $A_z = z$, this same result can be obtained by replacing into the material expression of the acceleration field the inverse equation of motion as follows.

$$\left.\begin{array}{l} y+z = (Y+Z)\,\mathrm{e}^{\,t} \\ y-z = (Y-Z)\,\mathrm{e}^{-t} \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} Y+Z = (y+z)\,\mathrm{e}^{-t} \\ Y-Z = (y-z)\,\mathrm{e}^{\,t} \end{array}\right.$$

$$\left\{\begin{array}{l} X = x \\ Y = \dfrac{1}{2}\left((y+z)\,\mathrm{e}^{-t} + (y-z)\,\mathrm{e}^{\,t}\right) \\ Z = \dfrac{1}{2}\left((y+z)\,\mathrm{e}^{-t} - (y-z)\,\mathrm{e}^{\,t}\right) \end{array}\right.$$

b) The material description of the acceleration in point $\mathbf{X}^* = (1,1,1)$ must be obtained. Replacing point $\mathbf{X}^*$ into the expression obtained in *a*) yields

$$\boxed{\mathbf{A}\,(\mathbf{X}^*,t) \overset{not}{\equiv} \left[0\,,\ \ \mathrm{e}^{\,t}\,,\ \ \mathrm{e}^{\,t}\right]^T}\,.$$

c) The difference between the spatial velocities per unit of time must be obtained, for point $\mathbf{x}^* = (1,1,1)$,

$$\frac{\Delta\mathbf{v}\,(\mathbf{x}^*,t)}{\Delta t} \quad \longrightarrow \quad \frac{\partial\mathbf{v}\,(\mathbf{x}^*,t)}{\partial t}\,.$$

The spatial description of the velocity field is

$$\mathbf{v}\,(\mathbf{x}=\mathbf{x}^*,\ t) = \mathbf{V}\,(\mathbf{X}\,(\mathbf{x}^*,t)\,,t)\ \,.$$

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Careful observation of the material expression of the velocity field obtained in *a*) reveals that $V_y = z$ and $V_z = y$, therefore

$$\mathbf{v}(\mathbf{x},t) \overset{not}{\equiv} [0\ ,\ \ z\ ,\ \ y]^T \quad \Longrightarrow \quad \boxed{\frac{\partial \mathbf{v}(\mathbf{x}^*,t)}{\partial t} \overset{not}{\equiv} [0\ ,\ \ 0\ ,\ \ 0]^T} .$$

Problem 1.4 – *Given the spatial description of the velocity field in Cartesian coordinates,*

$$\mathbf{v} \overset{not}{\equiv} [x\ ,\ \ y\ ,\ \ z\varphi(t)]^T$$

and the surface

$$\Sigma_t := \left\{ \mathbf{x} \mid F(x,y,z,t) = \mathrm{e}^{-2t}\left(x^2+y^2\right) + z^2\mathrm{e}^{-t^2} - C = 0 \right\} ,$$

where $C \neq 0$ is a constant, determine $\varphi(t)$ considering that the particles on this surface are always the same.

Solution

The function F defines the material surface $\Sigma_t := \{\mathbf{x} \mid F(x,y,z,t) = 0\}$. The necessary and sufficient condition for this surface to be a material surface is

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla F = 0 \qquad \forall \mathbf{x} \in \Sigma_t \quad \forall t ,$$

where

$$\frac{\partial F}{\partial t} = -2\mathrm{e}^{-2t}\left(x^2+y^2\right) - 2tz^2\mathrm{e}^{-t^2} ,$$

$$\nabla F \overset{not}{\equiv} \left[2x\mathrm{e}^{-2t}\ ,\ \ 2y\mathrm{e}^{-2t}\ ,\ \ 2z\mathrm{e}^{-t^2}\right]^T \ , \text{ and}$$

$$\mathbf{v}\cdot\nabla F = 2x^2\mathrm{e}^{-2t} + 2y^2\mathrm{e}^{-2t} + 2z^2\mathrm{e}^{-t^2}\varphi(t) .$$

Then, the necessary and sufficient condition above is reduced to

$$2z^2\left(\varphi(t) - t\right)\mathrm{e}^{-t^2} = 0 \qquad \forall \mathbf{x} \in \Sigma_t \quad \forall t .$$

Moreover, for $\mathbf{x} \in \Sigma_t$, the term z^2 can be isolated from the expression of the function defining the material surface $F(x,y,z,t)$ given in the statement, $z^2 = \left(C - \mathrm{e}^{-2t}\left(x^2+y^2\right)\right)\mathrm{e}^{t^2}$. Replacing this expression into the previous equation yields

$$2\left(C - \mathrm{e}^{-2t}\left(x^2+y^2\right)\right)\left(\varphi(t) - t\right) = 0 \qquad \forall \mathbf{x}\ \forall t .$$

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Since $\left(C - \mathrm{e}^{-2t}\left(x^2+y^2\right)\right) = 0$ cannot be satisfied for $\forall \mathbf{x}$ and $\forall t$ because C is a constant, the only possibility left is

$$\boxed{\varphi(t) = t}\,.$$

Problem 1.5 – *Given the velocity field of a perfect fluid*

$$\mathbf{v}(\mathbf{x},t) \overset{not}{\equiv} \left[z\mathrm{e}^{t}\,, \quad \frac{y}{1+t}\,, \quad \mathrm{v}_z\right]^T$$

and the surface $\varphi(\mathbf{x},t) = x - z(1+t)\,\mathrm{e}^{t} + k = 0$ *(where k is a constant), which is known to be a material surface, determine:*

a) The equation of the trajectory in canonical form and the equation of the streamlines.

b) The equation of the streakline and the position of its initial and final points if the spill point is $\mathbf{x}^*$ *and the spill period is* $t \in [t_1, t_2]$.

Solution

a) To be able to calculate the trajectories and streamlines, the expression for the velocity field must be completed. To find v_z, the information given about surface φ is used. The necessary and sufficient condition for this surface to be a material surface is

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \mathbf{v}\cdot\nabla\varphi = 0 \quad \forall \mathbf{x} \in \Sigma_t \qquad \forall t\,,$$

$$\text{where} \quad \frac{\partial\varphi}{\partial t} = -z\left(\mathrm{e}^{t} + \mathrm{e}^{t}(1+t)\right)\,, \quad \nabla\varphi \overset{not}{\equiv} \left[1\,,\ 0\,,\ -\mathrm{e}^{t}(1+t)\right]^T$$

$$\text{and} \quad \mathbf{v}\cdot\nabla\varphi = z\mathrm{e}^{t} - \mathrm{v}_z\mathrm{e}^{t}(1+t)\,.$$

Then, the material derivative of φ is

$$\frac{d\varphi}{dt} = -z\mathrm{e}^{t} - z\mathrm{e}^{t}(1+t) + z\mathrm{e}^{t} - \mathrm{v}_z\mathrm{e}^{t}(1+t) = 0$$

which results in $\mathrm{v}_z = -z$. Therefore, the spatial description of velocity field is

$$\mathbf{v}(\mathbf{x},t) \overset{not}{\equiv} \left[z\mathrm{e}^{t}\,, \quad \frac{y}{1+t}\,, \quad -z\right]^T\,.$$

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Now, this field must be integrated to obtain the equation of the trajectory since $d\mathbf{x}/dt = \mathbf{v}(\mathbf{x},t)$. Applying the equality for each component and particularizing for the velocity field determined yields

$$\frac{dx}{dt} = z\mathrm{e}^{t}\,, \quad \frac{dy}{dt} = \frac{y}{1+t} \quad \text{and} \quad \frac{dz}{dt} = -z\,.$$

Note that the x-component depends on the z-coordinate. Then, the z-coordinate must be determined first,

$$\frac{dz}{dt} = -z \quad \Longrightarrow \quad z = C_1\mathrm{e}^{-t}\,.$$

Replacing the expression found for z into the x-component and integrating the expression results in

$$\frac{dx}{dt} = C_1\mathrm{e}^{-t}\mathrm{e}^{t} = C_1 \quad \Longrightarrow \quad x = C_1 t + C_2\,.$$

Finally, the y-component is

$$\frac{dy}{dt} = \frac{y}{1+t} \quad \Longrightarrow \quad y = C_3(1+t)\,.$$

To obtain the canonical form of the expression, $\mathbf{x} = \mathbf{X}$ for $t = 0$ is imposed,

$$\begin{cases} x(0) = C_2 = X \\ y(0) = C_3 = Y \\ z(0) = C_1 = Z \end{cases}$$

and, finally, the equation of the trajectory in canonical form is

$$\boxed{\begin{aligned} x &= X + Zt \\ y &= Y(1+t) \\ z &= Z\mathrm{e}^{-t} \end{aligned}}\,.$$

The equation of the streamlines is found by integrating the velocity field with respect to λ, that is, $d\mathbf{x}(\lambda)/d\lambda = \mathbf{v}(\mathbf{x}(\lambda),t)$. As in the case of the equation of the trajectory, the z-component must be determined before the x-component,

$$\frac{dz}{d\lambda} = -z \quad \Longrightarrow \quad z = C_1\mathrm{e}^{-\lambda}\,.$$

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Replacing into the x-component yields

$$\frac{dx}{d\lambda} = C_1 \mathrm{e}^{\,(t-\lambda)} \quad \Longrightarrow \quad x = -C_1 \mathrm{e}^{\,(t-\lambda)} + C_2$$

and the remaining component results in

$$\frac{dy}{d\lambda} = \frac{y}{1+t} \quad \Longrightarrow \quad y = C_3 \mathrm{e}^{\,\frac{\lambda}{1+t}} \,.$$

Then, the equation of the streamlines is

$$\boxed{\begin{aligned} x &= -C_1 \mathrm{e}^{\,(t-\lambda)} + C_2 \\ y &= C_3 \mathrm{e}^{\,\frac{\lambda}{1+t}} \\ z &= C_1 \mathrm{e}^{-\lambda} \end{aligned}} \,.$$

b) To obtain the equation of the streakline it is enough to take the equation of motion and impose $\mathbf{x}^* = \mathbf{x}(\mathbf{X}, \tau)$, where τ is a time belonging to the spill period.

$$\begin{cases} x^* = X + Z\tau \\ y^* = Y(1+\tau) \\ z^* = Z\mathrm{e}^{-\tau} \end{cases}$$

And the inverse of this equation is

$$\begin{cases} X = x^* - Z\tau = x^* - z^*\tau\mathrm{e}^{\tau} \\ Y = \dfrac{y^*}{1+\tau} \\ Z = z^*\mathrm{e}^{\tau} \end{cases}$$

Replacing these into the equation of motion results in the equation of the streakline,

$$\boxed{\begin{aligned} x &= x^* - z^*(\tau - t)\mathrm{e}^{\tau} \\ y &= y^* \frac{1+t}{1+\tau} \\ z &= z^*\mathrm{e}^{(\tau-t)} \end{aligned}} \,.$$

Consider the physical concept of the streakline as the color line that would be observed in the medium if a tracer fluid were injected at the spill point through-

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out the spill period. Then, for each time t, the streakline can be visualized in terms of the parameter τ, which gives the position in space of the colored particles. It is verified that, as expected, $x = x^*$ for $t = \tau$, since it corresponds to the time in which the streakline is crossing the spill point. Now, the streakline must be delimited for each time t.

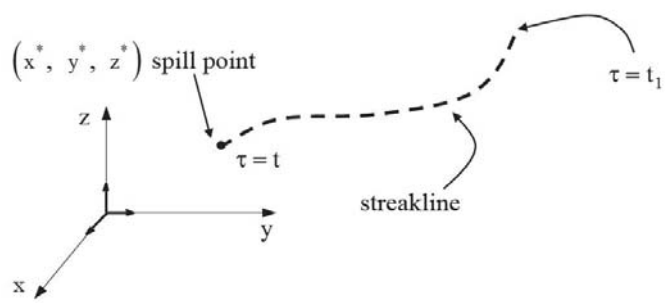
There are two distinct cases:

i) $\boxed{t_1 < t < t_2}$

The first colored point in the streakline is the one crossing the spill point at $\tau = t_1$ while the last one is the one crossing the spill point at $\tau = t$.

Initial point: $\begin{cases} x = x^* - z^*\,(t_1 - t)\,\mathrm{e}^{\,t_1} \\ y = y^* \dfrac{1+t}{1+t_1} \\ z = z^* \mathrm{e}^{\,(t_1 - t)} \end{cases}$ Final point: $\begin{cases} x = x^* \\ y = y^* \\ z = z^* \end{cases}$

$\boxed{t_1 < t < t_2}$



ii) $\boxed{t \geq t_2}$

The first colored point in the streakline is the same as in the previous case, $\tau = t_1$, but the last point is now $\tau = t_2$. The streakline has now "moved away" from the spill point.

Initial point: $\begin{cases} x = x^* - z^*\,(t_1 - t)\,\mathrm{e}^{\,t_1} \\ y = y^* \dfrac{1+t}{1+t_1} \\ z = z^* \mathrm{e}^{\,(t_1 - t)} \end{cases}$ Final point: $\begin{cases} x = x^* - z^*\,(t_2 - t)\,\mathrm{e}^{\,t_2} \\ y = y^* \dfrac{1+t}{1+t_2} \\ z = z^* \mathrm{e}^{\,(t_2 - t)} \end{cases}$

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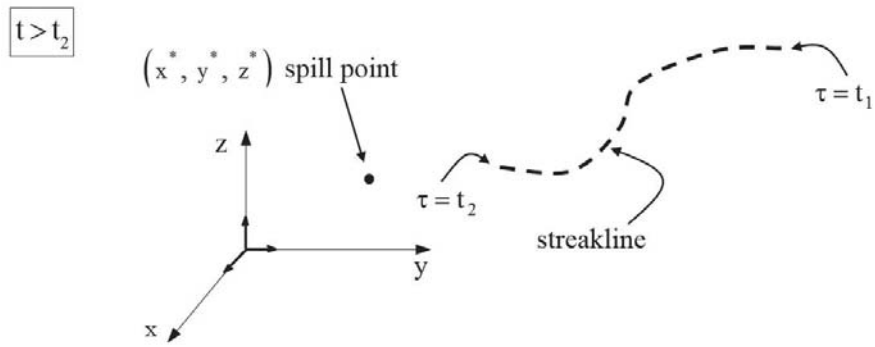

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EXERCISES

1.1 – *Justify if the following statements are true or false.*

a) Two streamlines corresponding to a same instant of time can never cross each other unless the velocity field at the cross point is zero.

b) Two different trajectories can never cross each other.

c) Two streaklines corresponding to two spill points with the same spill period can cross each other at one or more points.

1.2 – *Given the following velocity field in material description*

$$\mathbf{v} \overset{not}{\equiv} \left[A\mathrm{e}^{At}X_1, \quad BtX_1, \quad CX_3\right]^T,$$

with A, B and C constants, obtain its spatial description and the conditions A, B and C must fulfill for the motion to be feasible for $0 < t < \infty$.

1.3 – *Tracer fluid is injected at point* $(1,1,1)$ *of the interior of a fluid from time* $t = 1$ *to time* $t = 2$. *If the equation of the streamlines is*

$$x = C_1\mathrm{e}^{\lambda t} \; ; \quad y = C_2\mathrm{e}^{\lambda t} \; , \quad z = C_3\mathrm{e}^{2\lambda t}$$

determine the equation of the streakline, indicating its initial and final points for $t = 5$.

1.4 – *The spatial description of the velocity field of a fluid is*

$$\mathbf{v} \overset{not}{\equiv} \left[y\mathrm{e}^{-t}, \quad z\mathrm{e}^{t}, \quad 0\right]^T \; .$$

Tracer fluid is injected on plane $y = 0$ *at time* $t = 1$. *Obtain the spatial equation of the stain along time.*

1.5 – *A certain motion is defined by the velocity field*

$$\mathrm{v}_x = 2ax \quad ; \qquad \mathrm{v}_y = -by \quad ; \qquad \mathrm{v}_z = -\frac{z}{t+c} \; .$$

Determine:

a) The equation of the trajectory in canonical form and the equation of the streamlines.

b) The possible values of a, b and c such that the motion has physical sense for $t \in [0, \infty)$.

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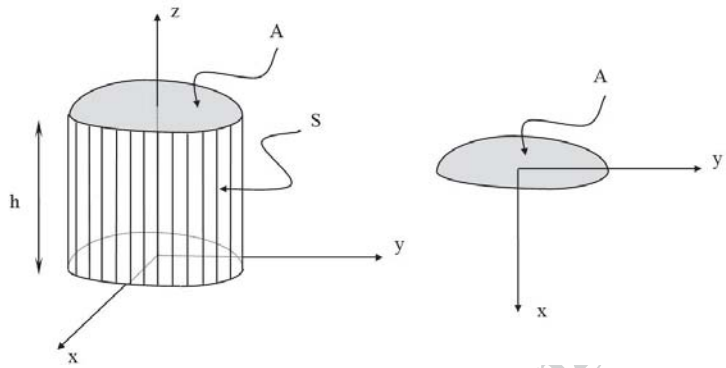
c) The spatial description of the material surface that, at time $t = 1$, was a sphere with center at $(0,0,0)$ and radius R (consider $a = b = c = 1$).



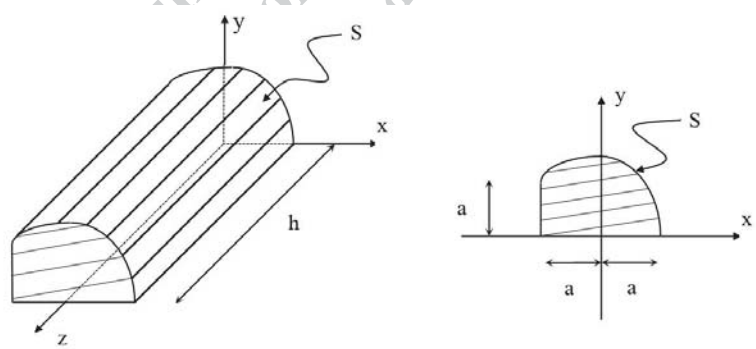
1.6 – *A certain motion is defined by the velocity field*

$$v_x = y e^{-t} \quad ; \qquad v_y = y \quad ; \qquad v_z = 0 \, .$$

Determine:

a) The equation of the trajectory in canonical form and the equation of the streamlines.

b) The spatial description of the material surface that, at time $t = 1$, was a sphere with center at $(0,0,0)$ and radius R.



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