Overview

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- Thermal Analogies
  - Solution to the linear thermoelastic problem
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6.1 Hypothesis of the Linear Elasticity Theory
Hypothesis of the Linear Elastic Model

- The simplifying hypothesis of the Theory of Linear Elasticity are:

  1. ‘Infinitesimal strains and deformation’ framework
  2. Existence of an unstrained and unstressed reference state
  3. Isothermal, isentropic and adiabatic processes
Hypothesis of the Linear Elastic Model

1. ‘Infinitesimal strains and deformation’ framework

- the displacements are infinitesimal:
  - material and spatial configurations or coordinates are the same
    \[ x = X + u \approx 0 \Rightarrow x \approx X \]
  - material and spatial descriptions of a property & material and spatial
differential operators are the same:
    \[ x = X \Rightarrow \gamma(x, t) = \gamma(X, t) = \Gamma(X, t) = \Gamma(x, t) \]
    \[ \frac{\partial (\bullet)}{\partial X} = \frac{\partial (\bullet)}{\partial x} \Rightarrow \nabla (\bullet) = \nabla (\bullet) \]
  - the deformation gradient \( F = \frac{\partial x}{\partial X} \approx 1 \Rightarrow |F| \approx 1 \), so the current spatial
density is approximated by the density at the reference configuration.
    \[ \rho_0 = \rho_t |F| \approx \rho_t \]

Thus, density is not an unknown variable in linear elastic problems.
Hypothesis of the Linear Elastic Model

1. ‘Infinitesimal strains and deformation’ framework
   - the displacement gradients are infinitesimal:
     - The strain tensors in material and spatial configurations collapse into the infinitesimal strain tensor:

\[ E(X,t) \approx e(x,t) = \varepsilon(x,t) \]
Hypothesis of the Linear Elastic Model

2. **Existence of an unstrained and unstressed reference state**
   - It is assumed that there exists a reference unstrained and unstressed neutral state, such that,
   
   \[ \varepsilon_0(x) = \varepsilon(x, t_0) = 0 \]
   
   \[ \sigma_0(x) = \sigma(x, t_0) = 0 \]

   - The reference state is usually assumed to correspond to the reference configuration.
3. Isothermal and adiabatic (=isentropic) processes

- In an **isothermal process** the temperature remains constant.
  \[ \theta(x, t) \equiv \theta(x, t_0) \equiv \theta_0(x) \quad \forall x \quad \Rightarrow \quad \dot{\theta} = 0 \]

- In an **isentropic process** the entropy of the system remains constant.
  \[ s(X, t) = s(X) = \frac{ds}{dt} = 0 \quad \Rightarrow \quad \dot{s} = 0 \]

- In an **adiabatic process** the net heat transfer entering into the body is zero.
  \[ Q_e = \int_{\Omega} \rho_0 \, r \, dV - \int_{\partial V} q \cdot n \, dS = 0 \quad \forall \Delta V \subset V \]

**Remark**

An **isentropic process** is an idealized thermodynamic process that is adiabatic, isothermal and **reversible**.
6.2 Linear Elastic Constitutive Equation

Ch. 6. Linear Elasticity
R. Hooke observed in 1660 that, for relatively small deformations of an object, the displacement or size of the deformation is directly proportional to the deforming force or load.

\[ F = k \Delta l \]

Hooke’s Law (for 1D problems) states that in an elastic material strain is directly proportional to stress through the elasticity modulus.

\[ \frac{F}{A} = E \frac{\Delta l}{l} \]

\[ \sigma = E \varepsilon \]
This proportionality is generalized for the multi-dimensional case in the Theory of Linear Elasticity.

\[
\begin{align*}
\sigma(x,t) &= \mathbf{C}(x) : \varepsilon(x,t) \\
\sigma_{ij} &= \mathbf{C}_{ijkl} \varepsilon_{kl} \quad i, j \in \{1, 2, 3\}
\end{align*}
\]

It constitutes the constitutive equation of a linear elastic material.

The 4\textsuperscript{th} order tensor \(\mathbf{C}\) is the constitutive elastic constants tensor:
- Has \(3^4 = 81\) components.
- Has the following symmetries, reducing the tensor to 21 independent components:
  - Minor symmetries: \(\mathbf{C}_{ijkl} = \mathbf{C}_{jikl}\)
  - Major symmetries: \(\mathbf{C}_{ijkl} = \mathbf{C}_{ijlk}\)

\[\mathbf{C}_{ijkl} = \mathbf{C}_{klij}\]

\textbf{REMARK}

The current stress at a point depends only on the current strain at the point, and not on the past history of strain states at the point.
The internal energy balance equation for the (adiabatic) linear elastic model is

\[ \frac{d}{dt} \int_V \rho_0 \ u \ dV = \int_V \frac{d\rho_0}{dt} \ u \ dV + \int_V \sigma : dV + \int_V (\rho_0 \ r - \nabla \cdot q) \ dV \]

Where:

- \( u \) is the specific internal energy (energy per unit mass).
- \( F \) is the specific heat generated by the internal sources.
- \( q \) is the heat conduction flux vector per unit surface.

Remark:

The rate of strain tensor is related to the material derivative of the material strain tensor through:

\[ \dot{E} = F^T \cdot d \cdot F \]

In this case, \( \dot{E} = \dot{\varepsilon} \) and \( F = I \).
Elastic Potential

- The stress power per unit of volume is an exact differential of the internal energy density, \( \dot{u} \), or internal energy per unit of volume:

\[
\frac{d}{dt}(\rho_0 u) = \frac{d\dot{u}(x,t)}{dt} = \dot{u} = \sigma : \dot{\varepsilon}
\]

- Operating in indicial notation:

\[
\frac{d\dot{u}}{dt} = \sigma : \dot{\varepsilon} = \dot{\varepsilon}_{ij} \sigma_{ij} = \dot{\varepsilon}_{ij} C_{ijkl} \varepsilon_{kl} = \frac{1}{2} \left( \dot{\varepsilon}_{ij} C_{ijkl} \varepsilon_{kl} + \dot{\varepsilon}_{ij} C_{ijkl} \varepsilon_{kl} \right) =
\]

\[
= \frac{1}{2} \left( \dot{\varepsilon}_{ij} C_{ijkl} \varepsilon_{kl} + \dot{\varepsilon}_{kl} C_{klij} \varepsilon_{ij} \right) = \frac{1}{2} \left( \dot{\varepsilon}_{ij} C_{ijkl} \varepsilon_{kl} + \varepsilon_{ij} C_{ijkl} \dot{\varepsilon}_{kl} \right) = \frac{d}{dt} (\varepsilon_{ij} C_{ijkl} \varepsilon_{kl})
\]

\[
= \frac{1}{2} \frac{d}{dt} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \rightarrow \frac{d\dot{u}}{dt} = \frac{d}{dt} \frac{1}{2} (\varepsilon : C : \dot{\varepsilon})
\]

**REMARK**

The symmetries of the constitutive elastic constants tensor are used:

- minor symmetries \( C_{ijkl} = C_{jikl} \)
- major symmetries \( C_{ijkl} = C_{ijlk} \)
Elastic Potential

\[ \frac{d\hat{U}}{dt} = \sigma : \dot{\varepsilon} = \frac{1}{2} \frac{d}{dt} (\varepsilon : C : \varepsilon) \]

Consequences:

1. Consider the time derivative of the internal energy in the whole volume:

\[ \int_V \frac{d}{dt} \hat{u}(x,t) \, dV = \frac{d}{dt} \int_V \hat{u}(x,t) \, dV = \frac{d}{dt} \hat{U}(t) = \int_V \sigma : \dot{\varepsilon} \, dV \]

- In elastic materials we talk about deformation energy because the stress power is an exact differential.

REMARC

The stress power, in elastic materials is an exact differential of the internal energy \( \hat{U}(t) \). Then, in elastic processes, we can talk of the elastic energy \( \hat{U}(t) \).
Elastic Potential

\[ \frac{d\hat{u}}{dt} = \sigma : \dot{\varepsilon} = \frac{1}{2} \frac{d}{dt} (\varepsilon : \mathbf{C} : \varepsilon) \]

- Consequences:
  2. Integrating the time derivative of the internal energy density,
     \[ \hat{u}(\mathbf{x},t) = \frac{1}{2} \varepsilon(\mathbf{x},t) : \mathbf{C} : \varepsilon(\mathbf{x},t) + a(\mathbf{x}) \]
     and assuming that the density of the internal energy vanishes at the neutral reference state, \( \hat{u}(\mathbf{x},t_0) = 0 \quad \forall \mathbf{x} \)

\[ \frac{1}{2} \varepsilon(\mathbf{x},t_0) : \mathbf{C} : \varepsilon(\mathbf{x},t_0) + a(\mathbf{x}) = a(\mathbf{x}) = 0 \quad \forall \mathbf{x} \quad \Rightarrow \quad \hat{u}(\varepsilon) = \frac{1}{2} \frac{\varepsilon : \mathbf{C} : \varepsilon}{\sigma} = \frac{1}{2} \frac{\sigma(\varepsilon) : \varepsilon}{\sigma} \]

- Due to thermodynamic reasons the internal energy is assumed always positive
  \[ \hat{u}(\varepsilon) = \frac{1}{2} \varepsilon : \mathbf{C} : \varepsilon > 0 \quad \forall \varepsilon \neq 0 \]
Elastic Potential

- The internal energy density defines a potential for the stress tensor, and is thus, named **elastic potential**. The stress tensor can be computed as

\[
\sigma(x,t) = C(x) : \varepsilon(x,t)
\]

- The constitutive elastic constants tensor can be obtained as the second derivative of the internal energy density with respect to the strain tensor field,

\[
\frac{\partial \sigma(\varepsilon)}{\partial \varepsilon} = \frac{\partial^2 \hat{u}(\varepsilon)}{\partial \varepsilon \otimes \partial \varepsilon} = \frac{\partial (C : \varepsilon)}{\partial \varepsilon} = C
\]

\[
C_{ijkl} = \frac{\partial^2 \hat{u}(\varepsilon)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}
\]
6.3 Isotropic Linear Elasticity
An isotropic elastic material must have the same elastic properties (contained in $\mathbf{C}$) in all directions.

- All the components of $\mathbf{C}$ must be independent of the orientation of the chosen (Cartesian) system → $\mathbf{C}$ must be a (mathematically) isotropic tensor.

$$\mathbf{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$$

$$\mathbf{C}_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \quad i, j, k, l \in \{1, 2, 3\}$$

Where:
- $\mathbf{I}$ is the 4th order unit tensor defined as $[\mathbf{I}]_{ijkl} = \frac{1}{2} \left[ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right]$.
- $\lambda$ and $\mu$ are scalar constants known as Lamé’s parameters or coefficients.

**REMARK**

The isotropy condition reduces the number of independent elastic constants from 21 to 2.
Introducing the isotropic constitutive elastic constants tensor $\mathbf{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$ into the generalized Hooke’s Law $\sigma = \mathbf{C} : \varepsilon$, in index notation:

$$\sigma_{ij} = \mathbf{C}_{ijkl} \varepsilon_{kl} = \left( \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right) \varepsilon_{kl} = \lambda \delta_{ij} \delta_{kl} \varepsilon_{kl} + 2\mu \left( \frac{1}{2} \delta_{ik} \delta_{jl} \varepsilon_{kl} + \frac{1}{2} \delta_{il} \delta_{jk} \varepsilon_{kl} \right) = \lambda \text{Tr}(\varepsilon) \delta_{ij} + 2\mu \varepsilon_{ij}$$

And the resulting constitutive equation is,

$$\begin{bmatrix} \sigma = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon \\
\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{ll} + 2\mu \varepsilon_{ij} \end{bmatrix}, \quad i, j \in \{1, 2, 3\}$$

Isotropic linear elastic constitutive equation. 

Hooke’s Law
Elastic Potential

If the constitutive equation is,

\[
\begin{aligned}
\sigma &= \lambda Tr(\varepsilon) \mathbf{1} + 2\mu \varepsilon \\
\sigma_{ij} &= \lambda \delta_{ij} \varepsilon_{ll} + 2\mu \varepsilon_{ij} \quad i, j \in \{1, 2, 3\}
\end{aligned}
\]

Then, the internal energy density can be reduced to:

\[
\hat{u}(\varepsilon) = \frac{1}{2} \sigma : \varepsilon = \frac{1}{2} \left( \lambda Tr(\varepsilon) \mathbf{1} + 2\mu \varepsilon \right) : \varepsilon = \sigma
\]

\[
= \frac{1}{2} \lambda Tr(\varepsilon) \mathbf{1} : \varepsilon + \frac{1}{2} 2\mu \varepsilon : \varepsilon
\]

\[
= \frac{1}{2} \lambda Tr^2(\varepsilon) + \mu \varepsilon : \varepsilon
\]

**Remark**

The internal energy density is an elastic potential of the stress tensor as:

\[
\frac{\partial \hat{u}(\varepsilon)}{\partial \varepsilon} = \sigma(\varepsilon) = \lambda Tr(\varepsilon) \mathbf{1} + 2\mu \varepsilon
\]
Inversion of the Constitutive Equation

1. $\varepsilon$ is isolated from the expression derived for Hooke’s Law
   \[ \sigma = \lambda Tr(\varepsilon) \mathbf{1} + 2\mu \varepsilon \quad \Rightarrow \quad \varepsilon = \frac{1}{2\mu} \left( \sigma - \lambda Tr(\varepsilon) \mathbf{1} \right) \]

2. The trace of $\sigma$ is obtained:
   \[ Tr(\sigma) = Tr(\lambda Tr(\varepsilon) \mathbf{1} + 2\mu \varepsilon) = \lambda Tr(\varepsilon) Tr(\mathbf{1}) + 2\mu Tr(\varepsilon) = (3\lambda + 2\mu) Tr(\varepsilon) \]
   \[ = 3 \]

3. The trace of $\varepsilon$ is easily isolated:
   \[ Tr(\varepsilon) = \frac{1}{3\lambda + 2\mu} Tr(\sigma) \]

4. The expression in 3. is introduced into the one obtained in 1.
   \[ \varepsilon = \frac{1}{2\mu} \left( \sigma - \lambda \frac{1}{3\lambda + 2\mu} Tr(\sigma) \mathbf{1} \right) \quad \Rightarrow \quad \varepsilon = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} Tr(\sigma) \mathbf{1} + \frac{1}{2\mu} \sigma \]
Inverse Isotropic Linear Elastic Constitutive Equation

- The Lamé parameters in terms of $E$ and $\nu$:
  \[
  E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}
  \]

- So the inverse const. eq. is re-written:
  \[
  \begin{align*}
  \varepsilon &= -\frac{\nu}{E} Tr(\sigma) \mathbf{1} + \frac{1+\nu}{E} \sigma \\
  \varepsilon_{ij} &= -\frac{\nu}{E} \sigma_{ii} \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij} \quad i, j \in \{1, 2, 3\}
  \end{align*}
  \]
  Inverse isotropic linear elastic constitutive equation. Inverse Hooke's Law.

In engineering notation:

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} \left(\sigma_x - \nu \left(\sigma_y + \sigma_z\right)\right) \\
\gamma_{xy} &= \frac{1}{G} \tau_{xy} \\
\varepsilon_y &= \frac{1}{E} \left(\sigma_y - \nu \left(\sigma_x + \sigma_z\right)\right) \\
\gamma_{xz} &= \frac{1}{G} \tau_{xz} \\
\varepsilon_z &= \frac{1}{E} \left(\sigma_z - \nu \left(\sigma_x + \sigma_y\right)\right) \\
\gamma_{yz} &= \frac{1}{G} \tau_{yz}
\end{align*}
\]
Young’s Modulus and Poisson’s Ratio

- **Young’s modulus** $E$ is a measure of the stiffness of an elastic material. It is given by the ratio of the uniaxial stress over the uniaxial strain.

  $$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

- **Poisson's ratio** $\nu$ is the ratio, when a solid is uniaxially stretched, of the transverse strain (perpendicular to the applied stress), to the axial strain (in the direction of the applied stress).

  $$\nu = \frac{\lambda}{2(\lambda + \mu)}$$
Example

Consider an uniaxial traction test of an isotropic linear elastic material such that:

\[ \sigma_x > 0 \]
\[ \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \]

Obtain the strains (in engineering notation) and comment on the results obtained for a Poisson’s ratio of \( \nu = 0 \) and \( \nu = 0.5 \).
\[ \sigma_x > 0 \]
\[ \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \]

For \( \nu = 0 \):

\[ \varepsilon_x = \frac{1}{E} \sigma_x \quad \gamma_{xy} = 0 \]
\[ \varepsilon_y = \frac{1}{E} \sigma_x \quad \gamma_{xz} = 0 \]
\[ \varepsilon_z = \frac{1}{E} \sigma_x \quad \gamma_{yz} = 0 \]

There is no Poisson’s effect and the transversal normal strains are zero.

For \( \nu = 0.5 \):

\[ \varepsilon_x = \frac{1}{E} \sigma_x \quad \gamma_{xy} = 0 \]
\[ \varepsilon_y = -\frac{0.5}{E} \sigma_x \quad \gamma_{xz} = 0 \]
\[ \varepsilon_z = -\frac{0.5}{E} \sigma_x \quad \gamma_{yz} = 0 \]

The volumetric deformation is zero, \( \text{tr}\, \varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z = 0 \), the material is incompressible and the volume is preserved.
Spherical and deviatoric parts of Hooke’s Law

- The stress tensor can be split into a spherical, or volumetric, part and a deviatoric part:

\[
\sigma_{sph} := \sigma_m \mathbf{1} = \frac{1}{3} \text{Tr}(\sigma) \mathbf{1} \\
\sigma' = \text{dev} \sigma = \sigma - \sigma_m \mathbf{1}
\]

- Similarly for the strain tensor:

\[
\varepsilon_{sph} = \frac{1}{3} e \mathbf{1} = \frac{1}{3} \text{Tr}(\varepsilon) \mathbf{1} \\
\varepsilon' = \text{dev} \varepsilon = \varepsilon - \frac{1}{3} e \mathbf{1}
\]
Spherical and deviatoric parts of Hooke’s Law

- Operating on the volumetric strain:
  \[ e = \text{Tr}(\varepsilon) \]
  \[ \varepsilon = -\frac{\nu}{E} \text{Tr}(\sigma) \begin{pmatrix} 1 & \frac{1+\nu}{E} \sigma \end{pmatrix} \]
  \[ e = -\frac{\nu}{E} \text{Tr}(\sigma) \begin{pmatrix} 1 \end{pmatrix} + \frac{1+\nu}{E} \text{Tr}(\sigma) \begin{pmatrix} 1 \end{pmatrix} = 3 \frac{1+\nu}{E} \text{Tr}(\sigma) = 3\sigma_m \]
  \[ e = \frac{3(1-2\nu)}{E} \sigma_m \rightarrow \sigma_m = \frac{E}{3(1-2\nu)} e \]

- The spherical parts of the stress and strain tensor are directly related:
  \[ \sigma_m = K e \]

\[ K \text{: bulk modulus (volumetric strain modulus)} \]
\[ K = \lambda + \frac{2}{3} \mu = \frac{E}{3(1-2\nu)} \]
Spherical and Deviator Parts of Hooke's Law

Introducing \( \sigma = \sigma_m \mathbf{1} + \sigma' \) into \( \varepsilon = -\frac{\nu}{E} \text{Tr}(\sigma) \mathbf{1} + \frac{1+\nu}{E} \sigma' \):

\[
\varepsilon = -\frac{\nu}{E} \text{Tr}(\sigma_m \mathbf{1} + \sigma') \mathbf{1} + \frac{1+\nu}{E} \left( \sigma_m \mathbf{1} + \sigma' \right) = -\frac{\nu}{E} \sigma_m \text{Tr} (\mathbf{1} + \sigma') + \frac{1+\nu}{E} \sigma' = \left( \frac{1+\nu}{E} - \frac{3\nu}{E} \right) \sigma_m \mathbf{1} + \frac{1+\nu}{E} \sigma'
\]

Taking into account that \( \sigma_m = \frac{E}{3(1-2\nu)} e \):

\[
\varepsilon = \left( \frac{1-2\nu}{E} \right) \frac{1}{3} \frac{E}{(1-2\nu)} e \mathbf{1} + \frac{1+\nu}{E} \sigma' = \frac{1}{3} e \mathbf{1} + \frac{1+\nu}{E} \sigma'
\]

Comparing this with the expression \( \varepsilon = \frac{1}{3} e \mathbf{1} + \varepsilon' \)

The deviatoric parts of the stress and strain tensor are related component by component:

\( \sigma' = 2G \varepsilon' \implies \sigma_{ij}' = 2G \varepsilon_{ij}' \quad i, j \in \{1, 2, 3\} \)
Spherical and deviatoric parts of Hooke’s Law

- The spherical and deviatoric parts of the strain tensor are directly proportional to the spherical and deviatoric parts (component by component) respectively, of the stress tensor:

\[ \sigma_m = K e \]

\[ \sigma'_{ij} = 2G \varepsilon'_{ij} \]
The internal energy density $\hat{u}(\varepsilon)$ defines a potential for the stress tensor and is, thus, an elastic potential:

$$\hat{u}(\varepsilon) = \frac{1}{2} \varepsilon : \mathbf{C} : \varepsilon$$

$\sigma = \frac{\partial \hat{u}(\varepsilon)}{\partial \varepsilon} = \mathbf{C} : \varepsilon$

Plotting $\hat{u}(\varepsilon)$ vs. $\varepsilon$:

There is a minimum for $\varepsilon = 0$:

$$\left. \frac{\partial \hat{u}(\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} = \left( \mathbf{C} : \varepsilon \right) \bigg|_{\varepsilon=0} = 0$$

$$\left. \frac{\partial^2 \hat{u}(\varepsilon)}{\partial \varepsilon \otimes \partial \varepsilon} \right|_{\varepsilon=0} = \left. \frac{\partial^2 \hat{u}(\varepsilon)}{\partial \varepsilon \otimes \partial \varepsilon} \right|_{\varepsilon=0} = \mathbf{C} \bigg|_{\varepsilon=0} = \mathbf{C}$$

REMARK

The constitutive elastic constants tensor $\mathbf{C}$ is positive definite due to thermodynamic considerations.
The elastic potential can be written as a function of the spherical and deviatoric parts of the strain tensor:

\[ \varepsilon : C : \varepsilon = (C : \varepsilon) : \varepsilon = \sigma : \varepsilon \]

\[ \hat{u}(\varepsilon) = \frac{1}{2} \varepsilon : C : \varepsilon = \frac{1}{2} \sigma : \varepsilon = \frac{1}{2} \left[ \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon \right] : \varepsilon = \sigma \]

\[ = \frac{1}{2} \lambda \text{Tr}(\varepsilon) \mathbf{1} : \varepsilon + \mu \varepsilon : \varepsilon = \frac{1}{2} \lambda \text{Tr}^2(\varepsilon) + \mu \varepsilon \varepsilon = \varepsilon^2 \]

\[ \hat{u}(\varepsilon) = \frac{1}{2} \lambda \varepsilon^2 + \frac{1}{3} \mu \varepsilon^2 + \mu \varepsilon' : \varepsilon' = \frac{1}{2} \left( \lambda + \frac{2}{3} \mu \right) \varepsilon^2 + \mu \varepsilon' : \varepsilon' = K \]

\[ \hat{u}(\varepsilon) = \frac{1}{2} K \varepsilon^2 + \mu \varepsilon' : \varepsilon' \geq 0 \]

Elastic potential in terms of the spherical and deviatoric parts of the strains.
The derived expression must hold true for any deformation process:

\[ \hat{u}(\varepsilon) = \frac{1}{2} K e^2 + \mu \varepsilon : \varepsilon \geq 0 \]

Consider now the following particular cases of isotropic linear elastic material:

- Pure spherical deformation process
  \[ \varepsilon^{(1)} = \frac{1}{3} e \mathbf{1} \]
  \[ \varepsilon^{(1)} = 0 \]

- Pure deviatoric deformation process
  \[ \varepsilon^{(2)} = \varepsilon' \]
  \[ e^{(2)} = 0 \]

\[ \hat{u}^{(2)} = \mu \varepsilon' : \varepsilon' \geq 0 \Rightarrow \mu > 0 \]

Lamé’s second parameter

**Remark**

\[ \varepsilon' : \varepsilon' = \varepsilon_{ij} \varepsilon_{ij} \geq 0 \]
Limits in the Elastic Properties

- $K$ and $\mu$ are related to $E$ and $\nu$ through:

$$K = \frac{E}{3(1-2\nu)} > 0$$
$$\mu = G = \frac{E}{2(1+\nu)} > 0$$

- Poisson’s ratio has a non-negative value,

$$\frac{E}{2(1+\nu)} > 0$$
$$\nu \geq 0$$

- Therefore,

$$\frac{E}{3(1-2\nu)} > 0$$
$$E \geq 0$$

$$0 \leq \nu \leq \frac{1}{2}$$

**REMARK**
In rare cases, a material can have a negative Poisson’s ratio. Such materials are named auxetic materials.
6.4 The Linear Elastic Problem

Ch.6. Linear Elasticity
The linear elastic solid is subjected to body forces and prescribed tractions:

The Linear Elastic problem is the set of equations that allow obtaining the evolution through time of the corresponding displacements $u(x,t)$, strains $\varepsilon(x,t)$ and stresses $\sigma(x,t)$. 
The Linear Elastic Problem is governed by the equations:

1. Cauchy’s Equation of Motion.
   Linear Momentum Balance Equation.
   \[ \nabla \cdot \sigma(x, t) + \rho_0 b(x, t) = \rho_0 \frac{\partial^2 u(x, t)}{\partial t^2} \]

2. Constitutive Equation.
   Isotropic Linear Elastic Constitutive Equation.
   \[ \sigma(x, t) = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon \]

   Kinematic Compatibility.
   \[ \varepsilon(x, t) = \nabla^S u(x, t) = \frac{1}{2} (u \otimes \nabla + \nabla \otimes u) \]

This is a PDE system of 15 eqns - 15 unknowns:

- \( u(x, t) \) 3 unknowns
- \( \varepsilon(x, t) \) 6 unknowns
- \( \sigma(x, t) \) 6 unknowns

Which must be solved in the \( \mathbb{R}^3 \times \mathbb{R}_+ \) space.
Boundary Conditions

- **Boundary conditions in space**
  - Affect the spatial arguments of the unknowns
  - Are applied on the boundary \( \Gamma \) of the solid, which is divided into three parts:
    - Prescribed displacements on \( \Gamma_u \):
      \[
      \begin{align*}
      u(x,t) &= u^*(x,t) \\
      u_i(x,t) &= u_i^*(x,t) \quad i \in \{1, 2, 3\}
      \end{align*}
      \]
    - Prescribed tractions on \( \Gamma_\sigma \):
      \[
      \begin{align*}
      \sigma(x,t) \cdot n &= t^*(x,t) \\
      \sigma_{ij}(x,t) \cdot n_j &= t_j^*(x,t) \quad i \in \{1, 2, 3\}
      \end{align*}
      \]
    - Prescribed displacements and stresses on \( \Gamma_{u\sigma} \):
      \[
      \begin{align*}
      u_i(x,t) &= u_i^*(x,t) \\
      \sigma_{jk}(x,t) \cdot n_k &= t_j^*(x,t) \quad (i, j, k \in \{1, 2, 3\}, \ i \neq j)
      \end{align*}
      \]
Boundary Conditions

\[ t_x^* = 0 \right\} \Gamma_\sigma \]
\[ t_y^* = 0 \right\} \Gamma_\sigma \]

\[ y \]
\[ x \]

\[ \Gamma_{u\sigma} \right\} \begin{cases} t_x^* = 0 \\ t_y^* = 0 \end{cases} \right\} \Gamma_\sigma \]
\[ u_x^* = 0 \]
\[ u_y^* = 0 \]

\[ \Gamma_\sigma \right\} \begin{cases} t_x^* = 0 \\ t_y^* = -P \end{cases} \right\} \Gamma_\sigma \]
\[ \Gamma_u \right\} \begin{cases} u_x^* = 0 \\ u_y^* = 0 \end{cases} \]
Boundary Conditions

- **Boundary conditions in time. INITIAL CONDITIONS.**
  - Affect the time argument of the unknowns.
  - Generally, they are the known values at $t = 0$:
    - Initial displacements:
      \[ u(x, 0) = 0 \quad \forall x \in V \]
    - Initial velocity:
      \[ \left. \frac{\partial u(x, t)}{\partial t} \right|_{t=0}^{not} = \dot{u}(x, 0) = v_0(x) \quad \forall x \in V \]
The Linear Elastic Problem

- Find the displacements $u(x,t)$, strains $\varepsilon(x,t)$, and stresses $\sigma(x,t)$ such that

$$\nabla \cdot \sigma(x,t) + \rho_0 b(x,t) = \rho_0 \frac{\partial^2 u(x,t)}{\partial t^2}$$
Cauchy’s Equation of Motion

$$\sigma(x,t) = \lambda Tr(\varepsilon) \mathbf{1} + 2\mu \varepsilon$$
Constitutive Equation

$$\varepsilon(x,t) = \nabla^S u(x,t) = \frac{1}{2}(u \otimes \nabla + \nabla \otimes u)$$
Geometric Equation

$$\Gamma_u : u = u^*$$
$$\Gamma_\sigma : t^* = \sigma \cdot n$$
Boundary conditions in space

$$u(x,0) = 0$$
$$\dot{u}(x,0) = v_0$$
Initial conditions (Boundary conditions in time)
The linear elastic problem can be viewed as a system of actions or data inserted into a mathematical model made up of the EDP’s and boundary conditions, which gives a response (or solution) in displacements, strains and stresses.

Generally, actions and responses depend on time. In these cases, the problem is a dynamic problem, integrated in $\mathbb{R}^3 \times \mathbb{R}_+$. In certain cases, the integration space is reduced to $\mathbb{R}^3$. The problem is termed quasi-static.
A problem is said to be \textbf{quasi-static} if the acceleration term can be considered to be negligible.

\[
a = \frac{\partial^2 u(x,t)}{\partial t^2} \approx 0
\]

This hypothesis is acceptable if actions are applied slowly. Then,

\[
\frac{\partial^2 A}{\partial t^2} \approx 0 \quad \Rightarrow \quad \frac{\partial^2 R}{\partial t^2} \approx 0 \quad \Rightarrow \quad \frac{\partial^2 u(x,t)}{\partial t^2} \approx 0
\]
Find the displacements $u(x,t)$, strains $\varepsilon(x,t)$, and stresses $\sigma(x,t)$ such that

$$\rho_0 \frac{\partial^2 u(x,t)}{\partial t^2} \approx 0$$

$\nabla \cdot \sigma(x,t) + \rho_0 b(x,t) = 0$

Equilibrium Equation

$$\sigma(x,t) = \lambda Tr(\varepsilon) \mathbf{1} + 2\mu \varepsilon$$

Constitutive Equation

$$\varepsilon(x,t) = \nabla^s u(x,t) = \frac{1}{2}(u \otimes \nabla + \nabla \otimes u)$$

Geometric Equation

$$\Gamma_u : \ u = u^*$$

$$\Gamma_\sigma : \ t^* = \sigma \cdot n$$

Boundary Conditions in Space

$$u(x,0) = 0$$

$$\dot{u}(x,0) = v_0$$

Initial Conditions
The quasi-static linear elastic problem does not involve time derivatives.

- Now the time variable plays the role of a loading descriptor: it describes the evolution of the actions.

**Mathematical model**

\[ \begin{align*}
    \mathbf{b}(x, \lambda) \\
    \mathbf{t}^*(x, \lambda) \\
    \mathbf{u}^*(x, \lambda)
\end{align*} \]

**EDPs+BCs**

\[ \begin{align*}
    \mathbf{u}(x, \lambda) \\
    \mathbf{\varepsilon}(x, \lambda) \\
    \mathbf{\sigma}(x, \lambda)
\end{align*} \]

**Actions**

\[ \mathbf{A}(x, \lambda) \]

**Responses**

\[ \mathbf{R}(x, \lambda) \]

- For each value of the actions \( \mathbf{A}(x, \lambda^*) \)-characterized by a fixed value \( \lambda^* \)- a response \( \mathbf{R}(x, \lambda^*) \) is obtained.

- Varying \( \lambda^* \), a family of actions and its corresponding family of responses are obtained.
Consider the typical material strength problem where a cantilever beam is subjected to a force $F(t)$ at its tip.

For a quasi-static problem,

$$\delta(\lambda) = \lambda \frac{F^* l^3}{3EI}.$$

The response is $\delta(t) = \delta(\lambda(\langle t \rangle))$, so for every time instant, it only depends on the corresponding value $\lambda(t)$. 
To solve the isotropic linear elastic problem posed, two approaches can be used:

- **Displacement formulation** - Navier Equations
  
  Eliminate $\sigma(x,t)$ and $\varepsilon(x,t)$ from the general system of equations. This generates a system of 3 eqns. for the 3 unknown components of $u(x,t)$.
  
  - Useful with displacement BCs.
  - Avoids compatibility equations.
  - Mostly used in 3D problems.
  - Basis of most of the numerical methods.

- **Stress formulation** - Beltrami-Michell Equations.
  
  Eliminates $u(x,t)$ and $\varepsilon(x,t)$ from the general system of equations. This generates a system of 6 eqns. for the 6 unknown components of $\sigma(x,t)$.
  
  - Effective with boundary conditions given in stresses.
  - Must work with compatibility equations.
  - Mostly used in 2D problems.
  - Can only be used in the quasi-static problem.
Displacement formulation

\[ \nabla \cdot \sigma(x, t) + \rho_0 b(x, t) = \rho_0 \frac{\partial^2 u(x, t)}{\partial t^2} \]  
Cauchy’s Equation of Motion

\[ \sigma(x, t) = \lambda Tr(\varepsilon) \mathbf{1} + 2\mu \varepsilon \]  
Constitutive Equation

\[ \varepsilon(x, t) = \nabla^s u(x, t) = \frac{1}{2} (u \otimes \nabla + \nabla \otimes u) \]  
Geometric Equation

\[ \Gamma_u : u = u^* \]
\[ \Gamma_\sigma : t^* = \sigma \cdot n \]  
Boundary Conditions in Space

\[ \begin{align*} 
 u(x, 0) &= 0 \\
 \dot{u}(x, 0) &= v_0 
\end{align*} \]  
Initial Conditions

The aim is to reduce this system to a system with \( u(x, t) \) as the only unknowns. Once these are obtained, \( \varepsilon(x, t) \) and \( \sigma(x, t) \) will be found through substitution.
Introduce the Constitutive Equation into Cauchy’s Equation of motion:

\[ \sigma(x,t) = \lambda \text{Tr}(\epsilon) \mathbf{1} + 2\mu \epsilon \]

\[ \nabla \cdot \sigma(x,t) + \rho_0 \mathbf{b}(x,t) = \rho_0 \frac{\partial^2 \mathbf{u}(x,t)}{\partial t^2} \]

Consider the following identities:

\[ \nabla \cdot (\text{Tr}(\epsilon) \mathbf{1}) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{3} \epsilon_{ij} \right) = \sum_{i=1}^{3} \left( \sum_{j=1}^{3} \frac{\partial \epsilon_{ij}}{\partial x_i} \right) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \nabla \cdot \mathbf{u} \right) = \nabla \cdot \mathbf{u} \]

\[ \nabla \cdot (\text{Tr}(\epsilon) \mathbf{1}) = \nabla (\nabla \cdot \mathbf{u}) \]
Introduce the Constitutive Equation into Cauchy’s Equation of motion:

\[ \sigma(x,t) = \lambda Tr(\varepsilon) \mathbf{1} + 2\mu \varepsilon \]

\[ \nabla \cdot \sigma(x,t) + \rho_0 b(x,t) = \rho_0 \frac{\partial^2 u(x,t)}{\partial t^2} \]

Consider the following identities:

\[ (\nabla \cdot \varepsilon)_i = \frac{\partial \varepsilon_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] = \frac{1}{2} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{2} \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_j} \right) = \frac{1}{2} \left( \nabla^2 u \right)_i + \frac{1}{2} \frac{\partial}{\partial x_i} (\nabla \cdot u) = \nabla \cdot u \]

\[ \nabla \cdot \varepsilon = \frac{1}{2} \nabla (\nabla \cdot u) + \frac{1}{2} \nabla^2 u \]
Introduce the Constitutive Equation into Cauchy’s Equation of Movement:

\[ \sigma(x,t) = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon \]

\[ \nabla \cdot \sigma(x,t) + \rho_0 b(x,t) = \rho_0 \frac{\partial^2 u(x,t)}{\partial t^2} \]

Replacing the identities:

\[ \nabla \cdot (\text{Tr}(\varepsilon) \mathbf{1}) = \nabla(\nabla \cdot u) \]

\[ \nabla \cdot \varepsilon = \frac{1}{2} \nabla(\nabla \cdot u) + \frac{1}{2} \nabla^2 u \]

Then, \( \lambda \nabla(\nabla \cdot u) + 2\mu \left( \frac{1}{2} \nabla(\nabla \cdot u) + \frac{1}{2} \nabla^2 u \right) + \rho_0 b = \rho_0 \frac{\partial^2 u}{\partial t^2} \)

The Navier Equations are obtained:

\[ \begin{aligned}
(\lambda + \mu) \nabla(\nabla \cdot u) + \mu \nabla^2 u + \rho_0 b &= \rho_0 \frac{\partial^2 u}{\partial t^2} \\
(\lambda + \mu) u_{j,i} + \mu u_{i,j} + \rho_0 b_i &= \rho_0 \dddot{u}_i \quad i \in \{1, 2, 3\}
\end{aligned} \]
Displacement formulation

- The boundary conditions are also rewritten in terms of $u(x,t)$:
  \[
  \sigma(x,t) = \lambda \text{Tr}(\varepsilon)\mathbf{1} + 2\mu \varepsilon
  \]
  \[
  t^* = \sigma \cdot n
  \]

- The BCs are now:

\[
\begin{cases}
  u = u^* \\
  u_i = u_i^* \quad i \in \{1, 2, 3\}
\end{cases}
\quad \text{on } \Gamma_u
\]

\[
\begin{cases}
  \lambda (\nabla \cdot u) n + \mu (u \otimes \nabla + \nabla \otimes u) \cdot n = t^* \\
  \lambda u_{k,k} n_i + \mu (u_{i,j} n_j + u_{j,i} n_j) = t_i^* \quad i \in \{1, 2, 3\}
\end{cases}
\quad \text{on } \Gamma_\sigma
\]

REMARK
The initial conditions remain the same.
Displacement formulation

- Navier equations in a **cylindrical coordinate system**:

\[
\begin{align*}
(\lambda + 2G) \frac{\partial e}{\partial r} - \frac{2G}{r} \frac{\partial \omega_z}{\partial \theta} + 2G \frac{\partial \omega_r}{\partial z} + \rho b_r &= \rho \frac{\partial^2 u_r}{\partial t^2} \\
(\lambda + 2G) \frac{1}{r} \frac{\partial e}{\partial \theta} - 2G \frac{\partial \omega_r}{\partial z} + 2G \frac{\partial \omega_z}{\partial r} + \rho b_\theta &= \rho \frac{\partial^2 u_\theta}{\partial t^2} \\
(\lambda + 2G) \frac{\partial e}{\partial z} - \frac{2G}{r} \frac{\partial \omega_r}{\partial r} (r \omega_\theta) + \frac{2G}{r} \frac{\partial \omega_z}{\partial \theta} + \rho b_z &= \rho \frac{\partial^2 u_z}{\partial t^2}
\end{align*}
\]

Where:

\[
\omega_r = -\Omega_{\theta z} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right)
\]

\[
\omega_\theta = -\Omega_{z r} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right)
\]

\[
\omega_z = -\Omega_{r \theta} = \frac{1}{2} \left( \frac{\partial (ru_\theta)}{r \partial r} - \frac{\partial u_r}{\partial \theta} \right)
\]

\[
e = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}
\]

\[
\mathbf{x}(r, \theta, z) \equiv \begin{cases} 
x = r \cos \theta \\
y = r \sin \theta \\
z = z
\end{cases}
\]

\[
dV = r \ d\theta \ dr \ dz
\]
Displacement formulation

- Navier equations in a spherical coordinate system:

\[
(\lambda + 2G) \frac{\partial e}{\partial r} - \frac{2G}{r \sin \theta} \frac{\partial}{\partial \theta} (\omega_\phi \sin \theta) + \frac{2G}{r \sin \theta} \frac{\partial \omega_\theta}{\partial \phi} + \rho b_r = \rho \frac{\partial^2 u_r}{\partial t^2} \\
(\lambda + 2G) \frac{\partial e}{\partial \theta} - \frac{2G}{r \sin \theta} \frac{\partial \omega_r}{\partial \phi} + \frac{2G}{r \sin \theta} \frac{\partial}{\partial r} (r \omega_\phi \sin \theta) + \rho b_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2} \\
(\lambda + 2G) \frac{\partial e}{\partial \phi} - \frac{2G}{r} \frac{\partial \omega_r}{\partial r} - \frac{2G}{r} \frac{\partial \omega_\theta}{\partial \theta} \frac{\partial}{\partial r} (r \omega_\phi \sin \theta) + \rho b_\phi = \rho \frac{\partial^2 u_\phi}{\partial t^2}
\]

Where:

\[
\omega_r = -\Omega_{\phi \theta} = \frac{1}{2} \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (u_\phi \sin \theta) - \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} \right) \\
\omega_\theta = -\Omega_{\phi r} = \frac{1}{2} \left( \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (ru_\phi) \right) \\
\omega_\phi = -\Omega_{r \theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial}{\partial r} (ru_{\theta}) - \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \theta} \right)
\]

\[
e = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} (r^2 u_r \sin \theta) + \frac{\partial}{\partial \theta} (ru_{\theta} \sin \theta) + \frac{\partial}{\partial \phi} (ru_\phi) \right]
\]

\[
x = x(r, \theta, \phi) = \begin{cases} 
  x = r \sin \theta \cos \phi \\
  y = r \sin \theta \sin \phi \\
  z = r \cos \theta
\end{cases}
\]

\[
dV = r^2 \sin \theta \, dr \, d\theta \, d\phi
\]
The aim is to reduce this system to a system with \( \sigma(x,t) \) as the only unknowns. Once these are obtained, \( \varepsilon(x,t) \) will be found through substitution and \( u(x,t) \) by integrating the geometric equations.

**REMARK**
For the quasi-static problem, the time variable plays the role of a loading factor.
Taking the geometric equation and, through successive derivations, the displacements are eliminated:

\[
\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \varepsilon_{jl}}{\partial x_i \partial x_k} = 0 \quad i, j, k, l \in \{1, 2, 3\}
\]

Compatibility Equations (seen in Ch.3.)

Introducing the inverse constitutive equation into the compatibility equations and using the equilibrium equation:

\[
\begin{align*}
\varepsilon_{ij} &= -\frac{\nu}{E} \sigma_{pp} \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij} \\
\frac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j &= 0
\end{align*}
\]

The Beltrami-Michell Equations are obtained:

\[
\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} = -\frac{\nu}{1-\nu} \delta_{ij} \left( \frac{\partial (\rho_0 b_k)}{\partial x_k} - \frac{\partial (\rho_0 b_i)}{\partial x_j} - \frac{\partial (\rho_0 b_j)}{\partial x_i} \right) \quad i, j \in \{1, 2, 3\}
\]
Stress formulation

- The **boundary conditions** are:
  - **Equilibrium Equations:** $\nabla \cdot \sigma + \rho_0 \mathbf{b} = 0$
    
    This is a 1st order PDE system, so they can act as boundary conditions of the (2nd order PDE system of the) Beltrami-Michell Equations
  
  - **Prescribed stresses on:** $\Gamma_{\sigma}$  
    $\sigma \cdot \mathbf{n} = \mathbf{t}^*$  on  $\Gamma_{\sigma}$
Stress formulation

- Once the stress field is known, the strain field is found by substitution.

\[ \varepsilon(x,t) = -\frac{\nu}{E} Tr(\sigma) \mathbf{1} + \frac{1+\nu}{E} \sigma \]

- The calculation, after, of the displacement field requires that the geometric equations be integrated with the prescribed displacements on \( \Gamma_u \):

\[ \begin{align*}
\varepsilon(x) &= \frac{1}{2} (u(x) \otimes \nabla + \nabla \otimes u(x)) \quad x \in V \\
u(x) &= u^*(x) \quad \forall x \in \Gamma_u
\end{align*} \]

**REMARK**

This need to integrate the second system is a considerable disadvantage with respect to the displacement formulation when using numerical methods to solve the linea! elastic problem.
From A. E. H. Love's *Treatise on the mathematical theory of elasticity*:

"According to the principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part."

Expressed in another way:

"The difference between the stresses caused by statically equivalent load systems is insignificant at distances greater than the largest dimension of the area over which the loads are acting."

**REMARK**

This principle does not have a rigorous mathematical proof.
Saint-Venant’s Principle

- Saint Venant’s Principle is often used in strength of materials.
- It is useful to introduce the concept of stress:

  The exact solution of this problem is very complicated.

This load system is statically equivalent to load system (I). The solution of this problem is very simple.

Saint Venant’s Principle allows approximating solution (I) by solution (II) at a far enough distance from the ends of the beam.
The solution of the lineal elastic problem is unique:

- It is unique in strains and stresses.
- It is unique in displacements assuming that appropriate boundary conditions hold in order to avoid rigid body motions.

This can be proven by Reductio ad absurdum ("reduction to the absurd"), as shown in pp. 189-193 of the course book.

- This proof is valid for lineal elasticity in infinitesimal strains.
- The constitutive tensor $C$ is used, so proof is not only valid for isotropic problems but also for orthotropic and anisotropic ones.
6.5 Linear Thermoelasticity
The simplifying hypothesis of the Theory of Linear Thermo-elasticity are:

1. Infinitesimal strains and deformation framework
   - Both the displacements and their gradients are infinitesimal.

2. Existence of an unstrained and unstressed reference state
   - The reference state is usually assumed to correspond to the reference configuration.
     \[ \varepsilon_0(x) = \varepsilon(x, t_0) = 0 \]
     \[ \sigma_0(x) = \sigma(x, t_0) = 0 \]

3. Isentropic and adiabatic processes – no longer isothermal !!!
   - Isentropic: entropy of the system remains constant
   - Adiabatic: deformation occurs without heat transfer
Hypothesis of the Linear Thermo-Elastic Model

3. (Hypothesis of isothermal process is removed)

- The process is no longer isothermal so the temperature changes throughout time:
  \[ \theta(x, t) \neq \theta(x, 0) = \theta_0 \]
  \[ \dot{\theta}(x, t) = \frac{\partial \theta(x, t)}{\partial t} \neq 0 \]

  We will assume the temperature field is known.

- But the process is still isentropic and adiabatic:
  \[ s(t) \equiv \text{cnt} \quad \Rightarrow \quad \dot{s} = 0 \]
  \[ Q_c = \int_V \rho \, r \, dV - \int_{\partial V} \mathbf{q} \cdot \mathbf{n} \, dS = 0 \quad \forall \Delta V \subset V \]

\[ \rho \, r - \nabla \cdot \mathbf{q} = 0 \quad \forall \mathbf{x} \quad \forall t \]
The Generalized Hooke’s Law becomes:

\[
\begin{align*}
\sigma(x,t) &= C : \varepsilon(x,t) - \beta(\theta - \theta_0) = C : \varepsilon(x,t) - \beta \Delta \theta, \\
\sigma_{ij} &= C_{ijkl} \varepsilon_{kl} - \beta_{ij} (\theta - \theta_0) \quad i, j \in \{1, 2, 3\}
\end{align*}
\]

Where

- \(C\) is the elastic constitutive tensor.
- \(\theta(x,t)\) is the absolute temperature field.
- \(\theta_0 = \theta(x,t_0)\) is the temperature at the reference state.
- \(\beta\) is the tensor of thermal properties or constitutive thermal constants tensor.

- It is a positive semi-definite symmetric second-order tensor.

**REMARK**

A symmetric second-order tensor \(A\) is positive semi-definite when \(z^T \cdot A \cdot z > 0\) for every non-zero column vector \(z\).
An isotropic thermoelastic material must have the same elastic and thermal properties in all directions:

- $C$ must be a (mathematically) isotropic 4th order tensor:

$$\begin{align*}
C &= \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I} \\
C_{ijkl} &= \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \\
&\quad \text{for } i, j, k, l \in \{1, 2, 3\}
\end{align*}$$

Where:

- $\mathbf{I}$ is the 4th order symmetric unit tensor defined as

$$[\mathbf{I}]_{ijkl} = \frac{1}{2} \left[ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right]$$

- $\lambda$ and $\mu$ are the Lamé parameters or coefficients.

- $\beta$ is a (mathematically) isotropic 2nd order tensor:

$$\begin{align*}
\mathbf{\beta} &= \beta \mathbf{1} \\
\beta_{ij} &= \beta \delta_{ij} \\
&\quad \text{for } i, j \in \{1, 2, 3\}
\end{align*}$$

Where:

- $\beta$ is a scalar thermal constant parameter.
Introducing the isotropic constitutive constants tensors $\beta = \beta \mathbf{1}$ and $\mathbf{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$ into the generalized Hooke’s Law, $\sigma = \mathbf{C} : \varepsilon - \beta (\theta - \theta_0)$ (in indicial notation)

$$
\sigma_{ij} = \mathbf{C}_{ijkl} \varepsilon_{kl} - \beta_{ij} (\theta - \theta_0) = 
\left( \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right) \varepsilon_{kl} - \beta (\theta - \theta_0) \delta_{ij} = 
\lambda \delta_{ij} \delta_{kl} \varepsilon_{kl} + 2\mu \left( \frac{1}{2} \delta_{ik} \delta_{jl} \varepsilon_{kl} + \frac{1}{2} \delta_{il} \delta_{jk} \varepsilon_{kl} \right) - \beta (\theta - \theta_0) \delta_{ij} = 
\varepsilon_{ll} - \beta (\theta - \theta_0) \delta_{ij} = \varepsilon_{ij} = \Delta \theta
$$

The resulting constitutive equation is,

$$
\begin{align*}
\sigma &= \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon - \beta \Delta \theta \mathbf{1} \\
\sigma_{ij} &= \lambda \delta_{ij} \varepsilon_{ll} + 2\mu \varepsilon_{ij} - \beta \Delta \theta \delta_{ij} \quad i, j \in \{1, 2, 3\}
\end{align*}
$$

Isotropic linear thermoelastic constitutive equation.
1. ε is isolated from the Generalized Hooke’s Law for linear thermoelastic problems:

\[ \sigma = C : \varepsilon - \beta \Delta \theta \quad \Rightarrow \quad \varepsilon = C^{-1} : \sigma + \Delta \theta \left( C^{-1} : \beta \right) \alpha \]

2. The \textbf{thermal expansion coefficients tensor} \( \alpha \) is defined as:

\[ \alpha \overset{\text{def}}{=} C^{-1} : \beta \]

It is a 2\textsuperscript{nd} order symmetric tensor which involves 6 thermal expansion coefficients.

3. The inverse constitutive equation is obtained:

\[ \varepsilon = C^{-1} : \sigma + \Delta \theta \alpha \]
For the isotropic case:

\[
\begin{align*}
\mathbf{C}^{-1} &= -\frac{\nu}{E} \mathbf{1} \otimes \mathbf{1} + \frac{1+\nu}{E} \mathbf{I} \\
\mathbf{C}_{ijkl}^{-1} &= -\frac{\nu}{E} \delta_{ij} \delta_{kl} + \frac{1+\nu}{E} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \quad i,j,k,l \in \{1,2,3\}
\end{align*}
\]

\[\rightarrow \alpha = \mathbf{C}^{-1} : (\beta \mathbf{1}) = \frac{1-2\nu}{E} \beta \mathbf{1}\]

The inverse const. eq. is re-written:

\[
\begin{align*}
\mathbf{\varepsilon} &= -\frac{\nu}{E} \text{Tr}(\mathbf{\sigma}) \mathbf{1} + \frac{1+\nu}{E} \mathbf{\sigma} + \alpha \Delta \theta \mathbf{1} \\
\varepsilon_{ij} &= -\frac{\nu}{E} \sigma_{ii} \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij} + \alpha \Delta \theta \delta_{ij} \quad i,j \in \{1,2,3\}
\end{align*}
\]

Inverse isotropic linear thermoelastic constitutive equation.

Where \(\alpha\) is a scalar **thermal expansion coefficient** related to the scalar thermal constant parameter \(\beta\) through:

\[\alpha = \frac{1-2\nu}{E} \beta\]
Comparing the constitutive equations,

\[ \sigma = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon \]

\[ \sigma = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon - \beta \Delta \theta \mathbf{1} \]

the decomposition is made:

\[ \sigma = \sigma^{nt} - \sigma^t \]

Where:

- \( \sigma^{nt} \) is the **non-thermal stress**: the stress produced if there is no temperature increment.
- \( \sigma^t \) is the **thermal stress**: the “corrector” stress due to the temperature increment.
Similarly, by comparing the inverse constitutive equations,

\[ \varepsilon = -\frac{\nu}{E} Tr(\sigma) \mathbf{1} + \frac{1+\nu}{E} \sigma \]

the decomposition is made:

\[ \varepsilon = \varepsilon^{nt} + \varepsilon' \]

Where:

- \( \varepsilon^{nt} \) is the **non-thermal strain**: the strain produced if there is no temperature increment.
- \( \varepsilon' \) is the **thermal strain**: the “corrector” strain due to the temperature increment.

Inverse isotropic linear **elastic** constitutive eq.

Inverse isotropic linear **thermoelastic** constitutive eq.
The thermal components appear when thermal processes are considered.

<table>
<thead>
<tr>
<th>TOTAL</th>
<th>NON-THERMAL COMPONENT</th>
<th>THERMAL COMPONENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = \sigma^{nt} - \sigma^t$</td>
<td>$\sigma^{nt} = C : \epsilon$</td>
<td>$\sigma^t = \Delta \theta \beta$</td>
</tr>
<tr>
<td>$\epsilon = \epsilon^{nt} + \epsilon^t$</td>
<td>$\epsilon^{nt} = \lambda \text{Tr}(\epsilon) \mathbf{1} + 2 \mu \epsilon$</td>
<td>Isotropic material: $\sigma^t = \beta \Delta \theta \mathbf{1}$</td>
</tr>
</tbody>
</table>

Isotropic material: $\sigma^{nt} = \lambda \text{Tr}(\epsilon) \mathbf{1} + 2 \mu \epsilon$  
Isotropic material: $\sigma^t = \beta \Delta \theta \mathbf{1}$

These are the equations used in FEM codes.

$$\begin{align*}
\sigma &= C : \epsilon^{nt} = C : [\epsilon - \epsilon^t] \\
\epsilon &= C^{-1} : \sigma^{nt} = C^{-1} : [\sigma + \sigma^t]
\end{align*}$$
Thermal Stress and Strain

REMARK 1
In thermoelastic problems, a state of zero strain in a body does not necessarily imply zero stress.

\[ \varepsilon = 0 \rightarrow \sigma^{nt} = 0 \]

\[ \sigma = -\sigma^t = -\beta \Delta \theta \mathbf{1} \neq 0 \]

REMARK 2
In thermoelastic problems, a state of zero stress in a body does not necessarily imply zero strain.

\[ \sigma = 0 \rightarrow \varepsilon^{nt} = 0 \]

\[ \varepsilon = \varepsilon^t = \alpha \Delta \theta \mathbf{1} \neq 0 \]

\[ \Delta \theta \neq 0 \]

\[ \varepsilon \neq 0 \]
6.6 Thermal Analogies

Ch.6. Linear Elasticity
Solution to the Linear Thermoelastic Problem

To solve the isotropic linear thermoelastic problem posed thermal analogies are used.

- The thermoelastic problem is solved like an elastic problem and then, the results are “corrected” to account for the temperature effects.

- They use the same strategies and methodologies seen in solving isotropic linear elastic problems:
  - Displacement Formulation - Navier Equations.
  - Stress Formulation - Beltrami-Michell Equations.

- Two basic analogies for solving quasi-static isotropic linear thermoelastic problems are presented:
  - 1st thermal analogy – Duhamel-Neumann analogy.
  - 2nd thermal analogy
1st Thermal Analogy

The governing eqns. of the quasi-static isotropic linear thermoelastic problem are:

\[ \nabla \cdot \sigma(x,t) + \rho_0 b(x,t) = 0 \quad \text{Equilibrium Equation} \]

\[ \sigma(x,t) = C : \varepsilon(x,t) - \beta \Delta \theta \mathbf{1} \quad \text{Constitutive Equation} \]

\[ \varepsilon(x,t) = \nabla^S u(x,t) = \frac{1}{2} (u \otimes \nabla + \nabla \otimes u) \quad \text{Geometric Equation} \]

\[ \Gamma_u : u = u^* \]

\[ \Gamma_\sigma : t^* = \sigma \cdot n \]

\[ \Gamma_n : u = u^* \]

\[ \Delta \theta \neq 0 \]

\[ b \]

\[ t^* \]
1st Thermal Analogy

- The actions and responses of the problem are:

\[ \text{ACTIONS} = \mathbb{A}^{(I)}(x, t) \]
\[ \text{RESPONSES} = \mathbb{R}^{(I)}(x, t) \]

**Elastic model**

EDPs+BCs

**REMARK**

\( \Delta \theta(x, t) \) is known a priori, i.e., it is independent of the mechanical response. This is an uncoupled thermoelastic problem.
To solve the problem following the methods used in linear elastic problems, the thermal term must be removed.

The stress tensor is split into $\sigma = \sigma^nt - \sigma^t$ and replaced into the governing equations:

**Momentum equations**

\[
\nabla \cdot \sigma = \nabla \cdot \sigma^{nt} - \nabla \cdot \sigma^t = \nabla \cdot \sigma^{nt} - \nabla (\beta \Delta \theta) \\
\beta \Delta \theta \mathbf{1}
\]

\[
\begin{align*}
\nabla \cdot \sigma + \rho_0 \mathbf{b} &= 0 \\
\nabla \cdot \sigma^{nt} + \rho_0 \left[ \mathbf{b} - \frac{1}{\rho_0} \nabla (\beta \Delta \theta) \right] &= 0
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot \sigma^{nt} + \rho_0 \hat{\mathbf{b}} &= 0 \\
\hat{\mathbf{b}} &= \mathbf{b} - \frac{1}{\rho_0} \nabla (\beta \Delta \theta)
\end{align*}
\]
Boundary equations:
\[
\sigma = \sigma^{nt} - \sigma^t
\]
\[
\sigma \cdot n = t^*
\]
\[
\sigma^{nt} \cdot n - \sigma^t \cdot n = t^*
\]
\[
\sigma^{nt} \cdot n = t^* + \sigma^t \cdot n = t^* + (\beta \Delta \theta)n
\]
\[
\beta \Delta \theta \cdot 1 \cdot n = t^*
\]
\[
\Gamma_{\sigma} : \sigma^{nt} \cdot n = t^*
\]
\[
\hat{t}^* = t^* + (\beta \Delta \theta)n
\]

ANALOGOUS PROBLEM — A linear elastic problem can be solved as:

\[
\nabla \cdot \sigma^{nt} + \rho_0 \dot{b} = 0 \quad \text{with} \quad \dot{b} = b - \frac{1}{\rho_0} \nabla (\beta \Delta \theta)
\]

\[
\sigma^{nt} = C : \varepsilon = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon
\]

\[
\varepsilon(x,t) = \nabla^S u(x,t) = \frac{1}{2} (u \otimes \nabla + \nabla \otimes u)
\]

\[
\Gamma_u : u = u^*
\]
\[
\Gamma_{\sigma} : \sigma^{nt} \cdot n = \hat{t}^* \quad \text{with} \quad \hat{t}^* = t^* + \beta \Delta \theta n
\]
1st Thermal Analogy

- The actions and responses of the **ANALOGOUS NON-THERMAL PROBLEM** are:

**ACTIONS** = \( \mathbf{A}^{(II)}(\mathbf{x}, t) \)

- \( \hat{\mathbf{b}}(\mathbf{x}, t) \)
- \( \hat{\mathbf{t}}^*(\mathbf{x}, t) \)
- \( \mathbf{u}^*(\mathbf{x}, t) \)

**RESPONSES** = \( \mathbf{R}^{(II)}(\mathbf{x}, t) \)

- \( \mathbf{u}(\mathbf{x}, t) \)
- \( \mathbf{\varepsilon}(\mathbf{x}, t) \)
- \( \mathbf{\sigma}^{nt}(\mathbf{x}, t) \)

**Elastic model**

- \( \mathbf{t}^* = \mathbf{t}^* + (\beta \Delta \theta) \mathbf{n} \)
- \( \mathbf{\Gamma}_{\sigma} : \mathbf{\sigma} \cdot \mathbf{n} = \mathbf{t}^* \)

**ANALOGOUS (ELASTIC) PROBLEM (II)**

**ORIGINAL PROBLEM (I)**

- \( \Delta \theta \neq 0 \)
- \( \mathbf{b} \)
- \( \mathbf{u} = \mathbf{u}^* \)

\[ \mathbf{t}^* = \mathbf{b} \quad \Delta \theta = 0 \]
\[ \mathbf{u} = \mathbf{u}^\circ \]
If the actions and responses of the original and analogous problems are compared:

**ACTIONS**

\[
A^{(I)}(x,t) - A^{(II)}(x,t) = \begin{pmatrix} b^* \\ u^* \\ t^* \end{pmatrix} - \begin{pmatrix} b^* \\ u^* \\ t^* \end{pmatrix} = \begin{pmatrix} b - b^* \\ 0 \\ t^* - t^* \end{pmatrix} = \begin{pmatrix} \nabla (\beta \Delta \theta) \\ 0 \\ - (\beta \Delta \theta) n \end{pmatrix}_{def} = A^{(III)}
\]

**RESPONSES**

\[
R^{(I)}(x,t) - R^{(II)}(x,t) = \begin{pmatrix} u \\ \varepsilon \\ \sigma \end{pmatrix} - \begin{pmatrix} u \\ \varepsilon \\ \sigma^{nt} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sigma - \sigma^{nt} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ - \beta \Delta \theta \mathbf{1} \end{pmatrix}_{def} = R^{(III)}
\]

Responses \( R^{(III)} \) are proven to be the solution of a thermoelastic problem under actions \( A^{(III)} \).
1st Thermal Analogy

THermoElastic
ORIGINAL
PROBLEM (I)

ANALOGOUS
ELASTIC
PROBLEM (II)

THERMOElastic
(TRIVIAL)
PROBLEM (III)

\[
\begin{align*}
\mathbf{A}^{(I)}(x,t) & = \mathbf{b}(x,t) \quad \mathbf{t}^*(x,t) \\
u^*(x,t) & = \Delta \theta(x,t) \\

\mathbf{R}^{(I)}(x,t) & = \begin{bmatrix} u(x,t) \\ \varepsilon(x,t) \\ \sigma(x,t) \end{bmatrix} \\

\mathbf{A}^{(II)}(x,t) & = \hat{\mathbf{b}}(x,t) = \mathbf{b} - \frac{1}{\rho_0} \nabla (\beta \Delta \theta) \\
\hat{\mathbf{t}}^*(x,t) & = \mathbf{t}^* + (\beta \Delta \theta) \mathbf{n} \\
u^*(x,t) & = \Delta \theta = 0 \\

\mathbf{R}^{(II)}(x,t) & = \begin{bmatrix} u(x,t) \\ \varepsilon(x,t) \\ \sigma(x,t) \end{bmatrix} \\

\mathbf{A}^{(III)}(x,t) & = \tilde{\mathbf{b}} = \frac{1}{\rho_0} \nabla (\beta \Delta \theta) \\
\tilde{\mathbf{t}}^* & = -(\beta \Delta \theta) \mathbf{n} \\
u^* & = 0 \\
\Delta \theta(x,t) & = 0 \\

\mathbf{R}^{(III)}(x,t) & = \begin{bmatrix} u(x,t) \\ \varepsilon(x,t) \\ \sigma(x,t) \end{bmatrix} = 0 \\
\varepsilon & = 0 \\
\sigma & = -\sigma' = -(\beta \Delta \theta) \mathbf{1}
\end{align*}
\]
The governing equations of the quasi-static isotropic linear thermoelastic problem are:

\[ \nabla \cdot \sigma(x,t) + \rho_0 \dot{b}(x,t) = 0 \quad \text{Equilibrium Equation} \]

\[ \varepsilon(x,t) = C^{-1} : \sigma(x,t) + \alpha \Delta \theta \mathbf{1} \quad \text{Inverse Constitutive Equation} \]

\[ \varepsilon(x,t) = \nabla^S u(x,t) = \frac{1}{2}(u \otimes \nabla + \nabla \otimes u) \quad \text{Geometric Equation} \]

\[ \Gamma_u : u = u^* \]

\[ \Gamma_\sigma : t^* = \sigma \cdot n \quad \text{Boundary Conditions in Space} \]
2nd Thermal Analogy

- The actions and responses of the problem are:

\[
\text{ACTIONS} = \mathbf{A}^{(I)}(x,t) \\
\text{RESPONSES} = \mathbf{R}^{(I)}(x,t)
\]

\[
\begin{align*}
\mathbf{b}(x,t) \\
\mathbf{t}^*(x,t) \\
\mathbf{u}^*(x,t) \\
\Delta \theta(x,t)
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}(x,t) \\
\mathbf{\varepsilon}(x,t) \\
\mathbf{\sigma}(x,t)
\end{align*}
\]

**Remark**

\(\Delta \theta(x,t)\) is known a priori, i.e., it is independent of the mechanical response. This is an uncoupled thermoelastic problem.
The assumption is made that $\Delta \theta(x,t)$ and $\alpha(x)$ are such that the thermal strain field $\varepsilon^t = \alpha \Delta \theta \mathbf{1}$ is integrable (satisfies the compatibility equations).

If the thermal strain field is integrable, there exists a field of thermal displacements, $u^t(x,t)$, which satisfies:

$$
\varepsilon^t(x,t) = (\alpha \Delta \theta) \mathbf{1} = \nabla^S u^t = \frac{1}{2} (u^t \otimes \nabla + \nabla \otimes u^t)
$$

$$
\varepsilon^t_{ij} = (\alpha \Delta \theta) \delta_{ij} = \frac{1}{2} \left( \frac{\partial u^t_i}{\partial x_j} + \frac{\partial u^t_j}{\partial x_i} \right) \hspace{10pt} i, j \in \{1,2,3\}
$$

**REMARK**

The solution $u^t(x,t)$ is determined except for a rigid body motion characterized by a rotation tensor $\Omega^*$ and a displacement vector $c^*$. The family of admissible solutions is $u^t(x,t) = \tilde{u}(x,t) + \Omega^* \cdot x + c^*$. This movement can be arbitrarily chosen (at convenience).

Then, the total displacement field is decomposed by defining:

$$
u^{nt}(x,t) \overset{\text{def}}{=} u(x,t) - u^t(x,t) \quad \Rightarrow \quad u = u^{nt} + u^t$$
To solve the problem following the methods used in linear elastic problems, the thermal terms must be removed.

The strain tensor and the displacement vector splits, \( \varepsilon = \varepsilon^{nt} + \varepsilon^t \) and \( u = u^{nt} + u^t \) are replaced into the governing equations:

**Geometric equations:**

\[
\varepsilon = \nabla^S u = \nabla^S (u^{nt} + u^t) = \nabla^S u^{nt} + \nabla^S u^t = \nabla^S u^{nt} + \varepsilon^t
\]

\[
\varepsilon^{nt} = \nabla^S u^{nt}
\]

**Boundary equations:**

\[
\begin{align*}
\Gamma_u : u &= u^* \\
\Gamma_u : u &= u^{nt} + u^t
\end{align*}
\]

\[
\Rightarrow u^{nt} + u^t = u^* 
\]

\[
\Rightarrow u^{nt} = u^* - u^t
\]
ANALOGOUS PROBLEM – A linear elastic problem can be solved as:

\[
\nabla \cdot \sigma + \rho_0 b = 0 \quad \text{Equilibrium Equation}
\]

\[
\varepsilon^{nt} = C^{-1} : \sigma \quad \text{Inverse constitutive Equation}
\]

\[
\varepsilon^{nt} = \nabla^S u^{nt} \quad \text{Geometric Equation}
\]

\[
\Gamma_u : u^{nt} = u^* - u^t \quad \text{Boundary Conditions in space}
\]

\[
\Gamma_\sigma : \sigma \cdot n = t^*
\]
The actions and responses of the analogous problem are:

**Actions**:
- $b(x,t)$
- $t^*(x,t)$
- $u^{nt} = u^*(x,t) - u'(x,t)$

**Responses**:
- $u^{nt}(x,t)$
- $\varepsilon^{nt}(x,t)$
- $\sigma(x,t)$

The 2nd Thermal Analogy

**Original Problem (I)**
- $\Delta \theta = 0$
- $\Gamma_u : u = u^*$

**Analogous Problem (II)**
- $\Delta \theta \neq 0$
- $\Gamma_\sigma : \sigma \cdot n = t^*$
- $\Gamma_u : u = u^* - u'$
If the actions and responses of the original and analogous problems are compared:

**ACTIONS**

\[
A^{(I)}(x,t) - A^{(II)}(x,t) = \begin{bmatrix} b \\ u^* \\ t^* \\ \Delta \theta \end{bmatrix} - \begin{bmatrix} b \\ u^*-u^t \\ t^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ u^t \\ 0 \\ \Delta \theta \end{bmatrix} = A^{(III)}
\]

**RESPONSES**

\[
R^{(I)}(x,t) - R^{(II)}(x,t) = \begin{bmatrix} u \\ \epsilon \\ \sigma \\ \Delta \theta \end{bmatrix} - \begin{bmatrix} u^{nt} \\ \epsilon^{nt} \\ \sigma \\ 0 \end{bmatrix} = \begin{bmatrix} u^t \\ \alpha \Delta \theta 1 \\ 0 \\ 0 \end{bmatrix} = R^{(III)}
\]

Responses \( R^{(III)} \) are proven to be the solution of a thermo-elastic problem under actions \( A^{(III)} \).
2nd Thermal Analogy

THERMOELASTIC ORIGINAL PROBLEM (I)

ANALOGOUS ELASTIC PROBLEM (II)

THERMOELASTIC TRIVIAL PROBLEM (III)

\[
\begin{align*}
A^{(I)}(x,t) & = b(x,t) \\
& \quad + t^*(x,t) \\
& = u^*(x,t) \\
& = \Delta \theta(x,t)
\end{align*}
\]

\[
\begin{align*}
R^{(I)}(x,t) & = u(x,t) \\
& \quad + \varepsilon(x,t) \\
& = \sigma(x,t)
\end{align*}
\]

\[
\begin{align*}
A^{(II)}(x,t) & = b \\
& = u^* - u' \\
& = t^* \\
& = \Delta \theta = 0
\end{align*}
\]

\[
\begin{align*}
R^{(II)}(x,t) & = u''(x,t) \\
& = \varepsilon''(x,t) \\
& = \sigma(x,t)
\end{align*}
\]

\[
\begin{align*}
A^{(III)}(x,t) & = \tilde{b} = 0 \\
& = \tilde{u}' = u'(x,t) \\
& = \tilde{t}' = 0 \\
& = \Delta \theta(x,t)
\end{align*}
\]

\[
\begin{align*}
R^{(III)}(x,t) & = u = u'(x,t) \\
& = \varepsilon' = (\alpha \Delta \theta) \mathbf{1} \\
& = \sigma = 0
\end{align*}
\]
2nd Analogy in structural analysis

\[ u_x = u_x' = \alpha \Delta \theta x \]
\[ \varepsilon_x = \varepsilon_x' = \alpha \Delta \theta \]
\[ \Gamma_u : u_x = u_x' \bigg|_{x=\ell} = \alpha \Delta \theta \ell \]

\[ u_x = u_x^{nt} \]
\[ \varepsilon_x = \varepsilon_x^{nt} \]
\[ \Gamma_u : u_x = u_x^* - u_x' \bigg|_{x=\ell} = -\alpha \Delta \theta \ell \]
Although the 2\textsuperscript{nd} analogy is more commonly used, the 1\textsuperscript{st} analogy requires less corrections.

The 2\textsuperscript{nd} analogy can only be applied if the thermal strain field is integrable.
- It is also recommended that the integration be simple.

The particular case
- Homogeneous material: $\alpha(x) = \text{const.} = \alpha$
- Lineal thermal increment: $\Delta \theta = ax + by + cz + d$

is of special interest because the thermal strains are:

$\varepsilon' \equiv \alpha \Delta \theta \mathbf{1} =$ linear polynomial

and trivially satisfy the compatibility conditions (involving second order derivatives).
In the particular case

- Homogeneous material: $\alpha(x) = \text{const.} = \alpha$
- Constant thermal increment: $\Delta \theta(x) = \text{const.} = \Delta \theta$

the integration of the strain field has a trivial solution because the thermal strains are constant $\varepsilon^t = \Delta \theta \alpha \mathbf{1} = \text{const.}$, therefore:

$$u^t(x, t) = \alpha \Delta \theta \mathbf{x} + \mathbf{\Omega}^* \cdot \mathbf{x} + \mathbf{c}^*$$

The thermal displacement is:

$$u^t(x, t) = \alpha \Delta \theta \mathbf{x} \quad \Rightarrow \quad \mathbf{x} + u^t = \mathbf{x} + \alpha \Delta \theta \mathbf{x} = (1 + \alpha \Delta \theta) \mathbf{x}$$

HOMOTHECY (free thermal expansion)
6.7 Superposition Principle

Ch.6. Linear Elasticity
Linear Thermoelastic Problem

- The governing eqns. of the isotropic linear thermoelastic problem are:

\[ \nabla \cdot \sigma(x,t) + \rho_0 b(x,t) = 0 \quad \text{Equilibrium Equation} \]

\[ \sigma(x,t) = C : \varepsilon(x,t) - \beta \Delta \theta \boldsymbol{1} \quad \text{Constitutive Equation} \]

\[ \varepsilon(x,t) = \nabla^S u(x,t) = \frac{1}{2} (u \otimes \nabla + \nabla \otimes u) \quad \text{Geometric Equation} \]

\[ \Gamma_u : \ u = u^* \]
\[ \Gamma_\sigma : \ t^* = \sigma \cdot n \quad \text{Boundary Conditions in space} \]

\[ u(x,0) = 0 \quad \text{Initial Conditions} \]
\[ \dot{u}(x,0) = \mathbf{v}_0 \]
Consider two possible systems of actions:

\[ A^{(1)}(x, t) \equiv \begin{align*}
&b^{(1)}(x, t) \\
&t^{*^{(1)}}(x, t) \\
&u^{*^{(1)}}(x, t) \\
&\Delta \theta^{(1)}(x, t) \\
&v_0^{(1)}(x)
\end{align*} \]

\[ A^{(2)}(x, t) \equiv \begin{align*}
&b^{(2)}(x, t) \\
&t^{*^{(2)}}(x, t) \\
&u^{*^{(2)}}(x, t) \\
&\Delta \theta^{(2)}(x, t) \\
&v_0^{(2)}(x)
\end{align*} \]

and their responses:

\[ R^{(1)}(x, t) \equiv \begin{align*}
&u^{(1)}(x, t) \\
&\varepsilon^{(1)}(x, t) \\
&\sigma^{(1)}(x, t)
\end{align*} \]

\[ R^{(2)}(x, t) \equiv \begin{align*}
&u^{(2)}(x, t) \\
&\varepsilon^{(2)}(x, t) \\
&\sigma^{(2)}(x, t)
\end{align*} \]
The solution to the system of actions $A^{(3)} = \lambda^{(1)} A^{(1)} + \lambda^{(2)} A^{(2)}$ where $\lambda^{(1)}$ and $\lambda^{(2)}$ are two given scalar values, is $R^{(3)} = \lambda^{(1)} R^{(1)} + \lambda^{(2)} R^{(2)}$.

The response to the lineal thermoelastic problem caused by two or more groups of actions is the lineal combination of the responses caused by each action individually.

This can be **proven by simple substitution** of the linear combination of actions and responses into the governing equations and boundary conditions.

When dealing with non-linear problems (plasticity, finite deformations, etc), this principle is no longer valid.
6.8 Hooke’s Law in Voigt Notation

Ch.6. Linear Elasticity
Taking into account the symmetry of the stress and strain tensors, these can be written in vector form:

\[
\sigma = \begin{bmatrix}
\varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\
\varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z
\end{bmatrix}
\]

\[
\epsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_{xy} \\
\varepsilon_{xz}
\end{bmatrix}
\]

\[
\sigma \epsilon = \sigma : \epsilon = \sigma_{ij} \varepsilon_{ij} = \sigma_i \varepsilon_i
\]

**Remark**

The double contraction \((\sigma : \epsilon)\) is transformed into the scalar (dot) product \((\{\sigma\} \cdot \{\epsilon\})\):

\[
\sigma = \begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_y & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_z
\end{bmatrix}
\]

\[
\epsilon = \begin{bmatrix}
\epsilon_x \\
\epsilon_{xy} \\
\epsilon_{xz}
\end{bmatrix}
\]

\[
\sigma \epsilon \equiv \{\sigma\} \cdot \{\epsilon\} = \sigma_x \epsilon_x + \sigma_y \epsilon_{xy} + \sigma_z \epsilon_{xz}
\]
Inverse Constitutive Equation

The inverse constitutive equation is rewritten:

\[ \varepsilon = -\frac{\nu}{E} \text{Tr}(\sigma) \mathbf{1} + \frac{1+\nu}{E} \sigma + \alpha \Delta \theta \mathbf{1} \]

\[ \{\varepsilon\} = \hat{\mathbf{C}}^{-1} \cdot \{\sigma\} + \{\varepsilon\}' \]

Where \( \hat{\mathbf{C}}^{-1} \) is an elastic constants inverse matrix and \( \{\varepsilon\}' \) is a thermal strain vector:

\[
\hat{\mathbf{C}}^{-1} = \begin{bmatrix}
\frac{1}{E} & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & \frac{1}{E} & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\]

\[ \{\varepsilon\}' = \begin{bmatrix}
\alpha \Delta \theta & 0 & 0 \\
0 & \alpha \Delta \theta & 0 \\
0 & 0 & \alpha \Delta \theta
\end{bmatrix} \]
Hooke’s Law

By inverting the inverse constitutive equation, Hooke’s Law in terms of the stress and strain vectors is obtained:

\[ \{\sigma\} = \hat{C} \cdot \left( \{\varepsilon\} - \{\varepsilon\}' \right) \]

Where \( \hat{C} \) is an elastic constants matrix:

\[
\hat{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\
\frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\
\frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \\
\end{bmatrix}
\]